1. Computability, Complexity and Algorithms

Part a: You are given a graph $G = (V, E)$ with edge weights $w(e) > 0$ for $e \in E$. You are also given a minimum cost spanning tree (MST) T. For one particular edge $e^* = (y, z)$ which is in T, its edge weight is increased (all other edges stay the same).
Specifically the weight of e^* changed from $w(e^*)$ to $\widehat{w}(e^*)$.

Give an algorithm to find a MST for this new edge weighting.

(As fast as possible in $O()$ notation.)

The graph G is given in adjacency list representation and the set T is given as a list of edges, for example as $\{4, 2\}, \{3, 4\}, \ldots.$ You are given the specified edge $e^* = (y, z)$ and you are given the old weights $w()$ for all edges of G, and the new weight $\widehat{w}(e^*)$.

Part b: Suppose you have a flow network $G = (V, E)$ with integer capacities $c_e > 0$ for $e \in E$, and you are given a maximum flow f^* from s to t. Let C^* denote the size of this flow f^*. Now suppose that for one particular edge e^* we decrease the capacity of e^* by one, from c_{e^*} to $c_{e^*} - 1$. Give an algorithm to output a maximum flow in the new graph. (As fast as possible in $O()$ notation.)

2. Analysis of Algorithms

Let $G = (V, E)$ be a directed graph with source s, sink t and capacities on edges. Give a polynomial time algorithm for deciding if G has a unique minimum s–t cut.

3. Theory of Linear Inequalities

Let $P \subseteq [0, 1]^n$ be an integral polytope contained in the 0/1 cube with ℓ_1-diameter bounded by k, i.e., the polytope's vertices have only entries in $\{0, 1\}$ and $\max_{x,y \in P} \|x - y\|_1 \leq k$. The goal is to maximize an objective $c \in \mathbb{Z}^n$ over P. Without loss of generality you may assume that $c \geq 0$ as we can simply flip the coordinates of the cube. You are given a feasible integral solution $x_0 \in P$ and access to the polytope P is restricted to querying the following oracle:

Augmentation oracle $O(x_0, c)$:
Input: $x_0 \in P$ integral, objective $c \in \mathbb{Z}^n$
Output: $x \in P$ integral with

$$cx > cx_0,$$

if such an x exists, otherwise return OPTIMAL.

Let $C := \|c\|_\infty$ and let $K := \lfloor \log C \rfloor$. Define the following sequence of objective functions $c^k := \lfloor c/2^{K-k} \rfloor$ (coordinate-wise operation) and consider the following bit scaling algorithm:

1. Repeat for $k = 0, \ldots, K$

   (a) While x_k is not $OPTIMAL$ for c^k do
      i. $x_k \leftarrow O(x_k, c^k)$
   (b) $x_{k+1} \leftarrow x_k$.

2. Return x_{K+1}.

Task.
Prove that the algorithm optimizes c over P with at most $(K+1)k$ oracle calls.

4. Combinatorial Optimization

We are given an undirected graph $G = (V, E)$ and every edge has a color. This is represented by a partition of E into $E_1 \cup \ldots \cup E_k$ where each E_i represents a set of edges of the same color i. A spanning tree is called bi-colorful if it contains at most two edges of any color.

1. Give an efficient algorithm that checks whether there is a a bi-colorful spanning tree in G and show its correctness.
2. Show that a graph G has a bi-colorful spanning tree if and only if for any disjoint set of colors $I, J \subseteq \{1, \ldots, k\}$ and any $F \subseteq \cup_{i \in I \cup J} E_i$ such that $|F \cap E_i| = 1$ for each $i \in I$ and $F \supseteq E_i$ for each $i \in J$, $G \setminus F$ has at most $|I| + 2|J| + 1$ components.

5. Graph Theory

Let $n \geq 1$ be an integer. At a round table there are $2n$ Canadians, $2n$ Americans and $2n$ Mexicans. The people whose neighbors are of the same nationality are asked to stand up. What is the largest possible number of people that can be asked to stand up?
Note: For example, for an American to stand up his two neighbors must be of the same nationality, but not necessarily American.

6. Probabilistic methods

Let $T = T(n, p)$ be the random (complete) binary tree of depth n (that is, it has 2^n leaves in total), where each edge is present with probability p. Let X_i be a random indicator variable for the reachability of the ith leaf from the root and denote by

$$X = \sum_{i=1}^{2^n} X_i$$

the number of reachable leaves from the root. Use the second moment method in order to show that for the threshold $p > 1/2$, one has

$$\mathbf{Prob}[X > 0] \ > \ 0.$$

Moreover, the lower bound on this probability is a constant that depends on p.
Hint: You can use the fact that $\mathbf{Prob}[X > 0] \geq \frac{(\mathbf{E}[X])^2}{\mathbf{E}[X^2]}$. (You do not need to prove this property.)

7. Algebra

Let ω be the complex number $\omega = e^{2\pi i/3}$ and let i be the complex number $i = e^{2\pi i/4}$. Which of the following rings are isomorphic?

1. $\mathbb{Z}[\omega]/ < 23 >$
2. $\mathbb{Z}[i]/ < 23 >$
3. $\mathbb{Z}/529$
4. $\mathbb{Z}/23 \times \mathbb{Z}/23$