1. Computability, Complexity and Algorithms

(a) Let $G(V, E)$ be an undirected unweighted graph. Let $C \subseteq V$ be a vertex cover of G. Argue that $V \setminus C$ is an independent set of G.
(b) Minimum cardinality vertex cover and maximum cardinality independent set are well known NP-complete problems. Suppose that you have a polynomial time approximation algorithm that, on input an undirected unweighted graph $G(V, E)$, outputs a vertex cover C whose cardinality is at most 2OPT. Is the cardinality of the independent set $V \setminus C$ a constant factor approximation algorithm for maximum independent set? If yes give a proof, if no give a counter example.
(c) Give a polynomial time algorithm that, on input an undirected unweighted bipartite graph G, outputs a minimum cardinality vertex cover of G.

2. Analysis of Algorithms

1. The *exact matching* problem is the following: Given a bipartite graph $G = (U, V, E)$ and an integer $k \leq n$, with $|U| = |V| = n$ and with a subset $E' \subset E$ of the edges colored red, an exact matching is a perfect matching with exactly k red edges. Give randomized polynomial-time algorithms for:

(a) Testing if G has an exact matching.

(b) If so find one.

2. Next consider an extension of this problem where two disjoint subsets E_1 and E_2 of edges are colored red and blue, respectively, and two integers k_1, k_2 are specified with $k_1 + k_2 \leq n$. Now we seek a perfect matching with k_1 red edges and k_2 blue edges. Repeat the two previous questions for this extended notion.

3. Theory of Linear Inequalities

For a system $Ax \leq b$ of m rational linear inequalities and a set $S \subseteq \{1, \ldots, m\}$ let

$$A_S x = b_S, \ A_{\bar{S}} x \leq b_{\bar{S}} \tag{1}$$

denote the system obtained by setting each inequality in S to equality while keeping each inequality in $\bar{S} = \{1, \ldots, m\} \setminus S$ as an inequality.

Suppose the system (1) has no solution for some specified set S. Then, by Farkas's Lemma, there exists a vector $(y_S, y_{\bar{S}})$ such that

$$y_S^T b_S + y_{\bar{S}}^T b_{\bar{S}} < 0, \ y_S^T A_S + y_{\bar{S}}^T A_{\bar{S}} = 0, \ y_{\bar{S}} \geq 0. \tag{2}$$

Due to the equality constraints, the vector y_S might have negative components. Notice, however, that we may scale $(y_S, y_{\bar{S}})$ so that it satisfies

$$y_S^T b_S + y_{\bar{S}}^T b_{\bar{S}} < 0,\ y_S^T A_S + y_{\bar{S}}^T A_{\bar{S}} = 0,\ y_S \geq -1,\ y_{\bar{S}} \geq 0. \tag{3}$$

Now (2) necessarily has an integral solution (since it has a rational solution), but (3) may not be solvable in integers. We say that the infeasibility of (1) can be *proven integrally* if (3) does in fact have an integral solution.

Prove the following theorem.

Theorem 1 *Let A be an integral matrix and let b be a rational vector such that $Ax \leq b$ has at least one solution. Then $Ax \leq b$ is totally dual integral if and only if*

(i) the rows of A form a Hilbert basis

and

(ii) for each subset S of inequalities from $Ax \leq b$, if (1) is infeasible, then this can be proven integrally.

4. Combinatorial Optimization

Recall that a graph G is *factor critical* if for all $v \in V(G)$, $G - v$ has a perfect matching. A *near perfect matching* is a matching covering all but one vertex of the graph. It is known that every 2-connected factor critical graph G contains pairwise edge-disjoint subgraphs $G_0, H_1, \ldots, H_k$ satisfying the following. For $j = 1, \ldots, k$, let $G_j = G_0 \cup \bigcup_{i=1}^{j} H_i$.

a. G_0 is an odd cycle and $G = G_k$,

b. H_i is an odd length path with both ends in G_{i-1} and no internal vertex in $V(G_{i-1})$. Specifically, the endpoints of H_i are distinct.

You may use this assertion without proof. Show that every 2-connected factor critical graph G contains at least $|E(G)|$ distinct near perfect matchings.

5. Graph Theory

Prove that for every integer $k \geq 1$ there exists an integer N such that if the subsets of $\{1, 2, \ldots, N\}$ are colored using k colors, then there exist disjoint non-empty sets $X, Y \subseteq \{1, 2, \ldots, N\}$ such that X, Y and $X \cup Y$ receive the same color.
Hint. You may want to consider intervals.

6. Probabilistic methods

A proper list-coloring of a graph $G = (V, E)$ from lists $\{L_v \subset \mathbb{N} \mid v \in V\}$ is a function $c : V \to \mathbb{N}$ such that $c\,(v) \in L_v$ for all $v \in V$ and $c\,(u) \neq c\,(v)$ for all $\{u, v\} \in E$.

Let r be a natural number. Prove that if for all $v \in V$ we have $|L_v| = 10r$ and for all $j \in L_v$ there are at most r neighbors $u \in V$ of v such that $j \in L_u$, then G admits a proper list-coloring from these lists.

7. Algebra

Two polynomials $f, g \in R[t]$ over a commutative ring R with identity are called *relatively prime over* R if f and g generate the unit ideal in $R[t]$. Let $f, g \in \mathbf{Z}[t]$ be non-constant monic polynomials such that f and g are relatively prime over $\mathbf{Q}$ and the residues of f and g modulo p are relatively prime over $\mathbf{Z}/p\mathbf{Z}$ for all prime numbers p. Prove that f and g are relatively prime over $\mathbf{Z}$.