## 1. Computability, Complexity and Algorithms

Let $G = (V, E)$ be a directed graph with integer weights $w_{ij}$ on its edges. The vertex set $V$ has been partitioned into $m$ subsets $V_1, \dots, V_m$, each containing $k$ vertices. Each vertex $u \in V_r$, for $1 \le r \le m-1$, has edges going to every vertex $v \in V_{r+1}$. These are all the edges of the graph.

1. Given two vertices $a \in V_1$ and $b \in V_m$ and an integer $W$, give an algorithm to determine whether there is a directed path from $a$ to $b$ of total weight at least $W$. The time complexity of your algorithm should be polynomial in $m$ and $k$.

2. Give a nondeterministic algorithm for the same problem that uses only $O(\log m + \log k)$ space.

3. Give a deterministic algorithm that uses $O((\log m + \log k)^2)$ space.

**Solution:** 1. Use a dynamic program to compute the maximum weight directed path between every pair of vertices $u \in V_i, v \in V_j$ for all $1 \le i < j \le m$. The corresponding table has $O(k^2m^2)$ entries and the computation of one entry (maximum length path for one pair of vertices) is done by using $k^2$ other entries in the table and $O(k^2)$ arithmetic operations. So the total time is $O(k^4m^2)$. The correctness is established by induction on the path length (or via the dynamic programming formula).
2. Guess the sequence of intermediate vertices between $a$ and $b$. Check that this sequence induces a path and has weight at least $W$. The number of vertices is $m$ and each takes $\log mk$ bits to write down.
3. Apply Savitch's simulation. Enumerate over possible choices for the vertex in the middle layer $V_{\lfloor r/2 \rfloor}$ on the path between $a$ and $b$. For each choice of $v$, find the maximum weight path between $a$ and $v$ and between $v$ and $b$ recursively. At any given point in this recursive enumeration, the algorithm has to store at most one vertex in each layer, for a total of $O((\log mk)^2)$ space.

## 2. Analysis of Algorithms

The 1-2 directed TSP is the following problem: Given a complete directed graph $G$ on $n$ vertices and with weights $w$ on its edges such that all $w(e) \in \{1, 2\}$, find a Hamilton cycle of minimum weight. Give a polynomial time 3/2 factor approximation algorithm for the 1-2 directed TSP. (Hint: Cover the vertices with cycles of minimum total weight, then patch the cycles.)

**Solution:** Attached on a separate page.

## 3. Theory of Linear Inequalities

Recall that given a polyhedron $P \subseteq \mathbb{R}^n$, $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, we have that

$$P_I \subseteq P^{\pi,\pi_0} := \operatorname{conv}\{(P \cap \{x \in \mathbb{R}^n \,|\, \pi x \le \pi_0\}) \cup (P \cap \{x \in \mathbb{R}^n \,|\, \pi x \ge \pi_0 + 1\})\},$$

where $P_I$ is the integer hull of $P$. The split closure $S(P)$ is defined as $S(P) := \bigcap_{\pi \in \mathbb{Z}^n, \pi_0 \in \mathbb{Z}} P^{\pi,\pi_0}$.

Suppose that $A \in \mathbb{Z}^{m \times n}$ has the property that on removing any column of $A$, the remaining matrix is totally unimodular. Prove that if $P := \{x \in \mathbb{R}^n \,|\, Ax \le b\}$ and $b \in \mathbb{Z}^m$, then $S(P) = P_I$.

**Solution.** We first claim that $S(P) \subseteq \bigcap_{\pi \in \mathbb{Z}^n} \operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, \pi x \in \mathbb{Z}\})$. It is sufficient to verify that $C := \bigcap_{\pi_0 \in \mathbb{Z}} P^{\pi,\pi_0} \subseteq \operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, \pi x \in \mathbb{Z}\}) =: D$ for a fixed $\pi \in \mathbb{Z}^n$. If $\hat{x} \in C$ and

$\pi\hat{x} \in \mathbb{Z}$, then clearly $\hat{x} \in D$. If $\hat{x} \in C$ and $\pi\hat{x} \notin \mathbb{Z}$, let $\hat{\pi}_0 = \lfloor \pi\hat{x} \rfloor$. Since $\hat{x} \in P^{\pi,\hat{\pi}_0}$, there exists $x^1 \in \{x \in \mathbb{R}^n \,|\, x \in P, \pi x \leq \hat{\pi}_0\}$ and $x^2 \in \{x \in \mathbb{R}^n \,|\, x \in P, \pi x \geq \hat{\pi}_0 + 1\}$ such that $x$ is a convex combination of $x^1$ and $x^2$. By updating $x^1$ by a suitable convex combination of $\hat{x}$ and $x^1$ (and similarly updating $x^2$ by a suitable convex combination of $\hat{x}$ and $x^2$), we may assume that $x \in \operatorname{conv}\{x^1, x^2\}$ where $x^1 \in \{x \in \mathbb{R}^n \,|\, x \in P, \pi x = \hat{\pi}_0\}$ and $x^2 \in \{x \in \mathbb{R}^n \,|\, x \in P, \pi x = \hat{\pi}_0 + 1\}$. Thus, we have that $\hat{x} \in D$.

$A$ has the property that on removing the first column of $A$, the remaining matrix is totally unimodular. Since $b$ is an integral vector, the above described property of $A$ implies that for any $\pi_0 \in \mathbb{Z}$ the polyhedron $\{x \in \mathbb{R}^n \,|\, Ax \leq b, x_1 = \pi_0\}$ is an integral polyhedron. Therefore $\operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, x_1 \in \mathbb{Z}\})$ is integral since it is the convex hull of integral polyhedra. Observe that $P_I \cap \mathbb{Z}^n = \operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, x_1 \in \mathbb{Z}\}) \cap \mathbb{Z}^n$. Therefore we conclude $P_I = \operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, x_1 \in \mathbb{Z}\})$. We now obtain that

$$
\begin{aligned}
P_I &= \operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, x_1 \in \mathbb{Z}\}) \\
&\supseteq \bigcap_{\pi \in \mathbb{Z}^n} \operatorname{conv}(\{x \in \mathbb{R}^n \,|\, x \in P, \pi x \in \mathbb{Z}\}) \\
&\supseteq S(P) \supseteq P_I,
\end{aligned}
$$

where the first inclusion is by the previous claim. Thus, $S(P) = P_I$.

## 4. Combinatorial Optimization

Observe that the Petersen graph has two perfect matchings, $\mathcal{M}_1$ and $\mathcal{M}_2$, such that $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect in exactly one edge. Equivalently, the matching $\mathcal{M}_2$ contains $\frac{2}{5}$ of the edges of $E(G) \setminus \mathcal{M}_1$.

Let $G$ be 3-regular graph with the property that for all $U \subseteq V(G)$ with $|U| \geq 2$, $|V(G) \setminus U| \geq 2$, we have that at least four edges of $G$ have one end in $U$ and the other end in $V(G) \setminus U$. Prove that for all perfect matchings $\mathcal{M}$ of $G$, there exists a matching $\mathcal{M}'$ such that $|\mathcal{M}' \cap (E(G) \setminus \mathcal{M})| \geq \frac{2}{5}|E(G) \setminus \mathcal{M}|$.

**Solution.** Let $G$ and $\mathcal{M}$ be given. First, observe that there does not exist a set $U \subseteq V(G)$, $|U| \geq 2$, $|V(G) \setminus U| \geq 2$ and $|U|$ odd such that $|\delta(U)| = 4$. Otherwise, by consolidating the vertex set $U$ to a single vertex, we would have a graph with an odd number of vertices of odd degree, a contradiction.

Consider the value $x \in \mathbb{R}^{E(G)}$ where $x_e = 1/5$ if $e \in \mathcal{M}$ and $x_e = 2/5$ otherwise. Then $x(\delta(v)) = 1$ for all $v \in V(G)$. Moreover, for any set $U \subseteq V(G)$, $|U|$ odd with $|U| \geq 3$, $|V(G) \setminus U| \geq 3$, we have that $x(\delta(U)) \geq 1$ since $|\delta(U)| \geq 5$. We conclude that $x$ is in the perfect matching polytope of $G$.

Since $x$ is contained in the perfect matching polytope of $G$, there exist perfect matchings $\mathcal{M}_1, \ldots, \mathcal{M}_k$ for some positive integer $k$ such that if we let $x_i$ be the indicator vector of $\mathcal{M}_i$, then $x$ is equal to the convex combination $\sum_1^n \alpha_i x_i$. Thus,

$$\sum_1^n \alpha_i |\mathcal{M}_i \setminus \mathcal{M}| = \sum_{e \in E(G) \setminus \mathcal{M}} x(e) = \frac{2}{5}|E(G) \setminus \mathcal{M}|.$$

We conclude that there exists an index $i$ such that $\mathcal{M}_i$ contains at least $\frac{2}{5}$ of the edges of $E(G) \setminus \mathcal{M}$, as desired.

## 5. Graph Theory

Let $G$ be a simple 3-regular graph, and let $k$ be its edge-chromatic number. Prove that if every two $k$-edge-colorings of $G$ differ by a permutation of colors, then $k = 3$ and $G$ has three distinct Hamiltonian cycles.

**Solution:** By Vizing's theorem $k = 3$ or $k = 4$. Suppose for a contradiction that $k = 4$, and let $f : E(G) \to \{1, 2, 3, 4\}$ be a proper edge-coloring. Let $H_{12}$ be the subgraph of $G$ induced by edges $e$ such that $f(e) \in \{1, 2\}$. Then $H_{12}$ has maximum degree at most two, and it is a spanning subgraph, because every vertex is incident with an edge colored 1 or 2. Furthermore, $H_{12}$ is connected, because otherwise swapping the colors 1 and 2 on one component of $H_{12}$ would produce a $k$-edge-coloring that cannot be obtained from $f$ by permuting colors. Thus $H_{12}$ is a Hamilton path or Hamilton cycle. The same applies to the analogously defined graph $H_{34}$. However, $H_{12}$ and $H_{34}$ are edge-disjoint, and hence $G$ has at most four vertices, contrary to the fact that $k = 4$.

Thus $k = 3$. Let us consider an arbitrary 3-edge-coloring of $G$. The union of every two color classes is a Hamiltonian cycle by the same argument as above. Thus $G$ has three distinct Hamiltonian cycles, as required.

## 6. Probabilistic methods

Show that there exists an absolute constant $c$ so that if $\{S_i : 1 \le i \le n\}$ is any sequence of sets with $|S_i| \ge c$, for all $i = 1, 2, \dots, n$, then there exists a sequence $\{x_i : 1 \le i \le n\}$ with $x_i \in S_i$, for all $i = 1, 2, \dots, n$, which is square-free, i.e., there is no pair $i, j$ with $1 \le i < j \le 2j - i - 1 \le n$ so that $x_{i+k} = x_{j+k}$ for all $k = 0, 1, \dots, j - i - 1$. Hint: This is an application of the asymmetric version of the Lovasz Local Lemma.

**Solution:** Clearly, we may assume $n$ is very large. To see, this simply expand the list of sets by adding arbitrary $c$ elements sets. Any initial portion of a square-free string is square-free.

Now suppose that each set $S_i$ has $c$ elements (as usual $c$ will be specified later). Then we form a word $x_1x_2x_3 \dots x_n$ by making a random choice from each $S_i$ with all elements of $S_i$ being equally likely. For each pair $(i, k)$ with $1 \le i < i + 2k - 1 \le n$, let $A(i, k)$ be the event that the length $k$ substring $x_i x_{i+1} \dots x_{i+k-1}$ is the first half of a square and is repeated in positions $x_{i+k}x_{i+k+1} \dots x_{i+2k-1}$.

Since the characters in the string are chosen at random, we note that $\Pr[A(i, k)] \le 1/c^k$.

Clearly, the dependency neighborhood of $A(i, k)$ consists on those events $A(j, m)$ where $[i, i + 2k - 1] \cap [j, j + 2m - 1] \ne \emptyset$. So we group them according to the value of $m$. For each value of $m$, there are (at most) $2k + 2m - 1$ such events.

To apply the Local Lemma, we will set $x(i, k) = 1/d^k$ where $d$ will be a constant depending on $c$ and just a bit smaller. Now the inequality we need is:

$$\frac{1}{c^k} \le \frac{1}{d^k} \prod_{m=1}^{n/2} (1 - \frac{1}{d^m})^{2k+2m-1}.$$

Multiplying both sides by $d^k$ and taking logarithms, the preceding inequality becomes

$$k \ln(d/c) \le \sum_{m=1}^{n/2} (2k + 2m - 1) \ln(1 - \frac{1}{d^m}).$$

Recall that when $|x| < 1$,

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$

Taking derivatives we have

$$\frac{1}{(1-x)^2} = \sum_{m=1}^{\infty} m x^{m-1}.$$

We use these two formulas, the approximation $\ln(1 - 1/d^m)$ by $-1/d^m$ and multiply both sides by $-1$, to obtain:

$$\begin{aligned} k\ln(c/d) &\geq \sum_{m=1}^{n/2} (2k + 2m - 1)\frac{1}{d^m} \\ &\sim \frac{2k-1}{d}\sum_{m=0}^{\infty}\frac{1}{d^m} + \frac{2}{d}\sum_{m=1}^{\infty} m\frac{1}{d^{m-1}} \\ &= \frac{2k-1}{d}\frac{d}{d-1} + \frac{2}{d}\frac{d^2}{(d-1)^2} \end{aligned}$$

Now it is easy to see that suitable choices for $c$ and $d$ can be found.

## 7. Algebra

Let $n$ be an integer $> 1$ and let $S_n$ be the symmetric group on the set $\{1, \ldots, n\}$. Let $G \subset S_n$ be a subgroup which acts transitively on $\{1, \ldots, n\}$. Prove that there is an element $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in \{1, \ldots, n\}$. Give a complete proof using only basic facts about groups acting on sets.

**Solution:** For $a \in \{1, \ldots, n\}$, let $G_a$ be the stabilizer of $a$:

$$G_a = \{\sigma \in G \mid \sigma(a) = a\}.$$

An element $\sigma$ satisfies $\sigma(a) \neq a$ if and only if $\sigma \notin G_a$, so we have to show that there is an element in $G$ not in any $G_a$, i.e., that

$$G \neq \cup_{a\in\{1,\ldots,n\}} G_a.$$

Note also that if $\sigma(a) = b$, then $G_b = \sigma G_a \sigma^{-1}$ (the conjugate group). Because $G$ acts transitively, every $G_a$ is $\sigma G_1 \sigma^{-1}$ for some $\sigma$, so it will suffice to show that

$$G \neq \cup_{\sigma\in G} \sigma G_1 \sigma^{-1}.$$

This follows from the general fact (proven below) that if $G$ is a finite group and $H$ is a proper subgroup, then $G$ is not the union of conjugates of $H$. Indeed, our $G$ is finite by assumption, and since $n > 1$ and $G$ acts transitively, $G_1 \neq G$.

To prove the general fact, let $h = |H|$ and $m = [G : H]$, so that $|G| = mh$. We consider the action of $G$ on itself by conjugation. The stabilizer of $H$ contains $H$, so has index at most $m$ in $G$. This means that there are at most $m$ distinct subgroups of the form $gHg^{-1}$. Each of them has order $h$, and they all contain the identity element, so the cardinality of $\cup_{g\in G} gHg^{-1}$ is at most $m(h-1) + 1 = mh - m + 1$. Since $H$ is a proper subgroup, $m > 1$, so this count is strictly less than $mh$, the order of $G$.

**Solution to problem 2:**

Below we show how to find a minimum weight cycle cover in polynomial time. Suppose that we have such a minimum weight cycle cover $C^*$ and let $H^*$ be a minimum weight Hamilton cycle. Since $H^*$ is also a cycle cover, we have $w(C^*) \leq w(H^*)$. Also $n \leq w(H^*)$ since all edge weights are $\geq 1$. We will repeatedly patch together two cycles to create a new cycle. Since every cycle has at least 2 vertices, we will need at most $\lfloor n/2 \rfloor - 1$ patching steps. We will show that we can do this in such a way that the total weight of the cycles increases by at most $\lfloor n/2 \rfloor$. Then we will end up with a Hamilton cycle of weight:

$$w(C^*) + \left\lfloor \frac{n}{2} \right\rfloor \leq w(H^*) + \frac{n}{2} \leq w(H^*) + \frac{w(H^*)}{2} = \frac{3}{2} w(H^*).$$

That leaves two things to explain: how to patch the cycles while not increasing the weight too much, and how to find the minimum cost cycle cover.

Patching the cycles: We patch two cycles by removing one edge from each, say $e_1$ and $e_2$, and adding two new edges to create a larger cycle, say $e_3$ and $e_4$ (which is always possible because $G$ is a complete directed graph.) The only way that this could increase the total weight by more than 1 is if both $e_1$ and $e_2$ had weight 1 while both $e_3$ and $e_4$ have weight 2; the total weight would then be increased by 2. We can avoid this as long as we choose $e_1$ to have weight 2, which is possible when there is some cycle in the present cycle cover that has some edge with weight 2. The only case where we cannot avoid this is when all edges in the present cycle cover have weight 1. So we might have to increase the weight by 2 in one patching step. But, after that, there will be an edge of weight 2. So we can avoid this from then on, until we return to a situation where all edges in the present cycle cover have weight 1. However, returning to a situation where all edges in the present cycle cover have weight 1 can only happen if in the previous step the total weight was decreased. Therefore, we can guarantee that after $k$ patching steps the total weight will increase by at most $k + 1$. Now, since we have $\lfloor n/2 \rfloor - 1$ patching steps, the total increase in weight will be at most $\lfloor n/2 \rfloor$ .

Minimum weight cycle cover: Where $G(V, E)$ is a complete directed weighted graph, we construct a bipartite graph $G'(X, Y, E')$ with weights on its edges as follows. For every vertex $v \in V$ we add one vertex $v_x \in X$ and another vertex $v_y \in Y$. For every directed edge $e = (v, u) \in E$ of weight *w(e)* we add one edge $e' = \{v_x, u_y\} \in E'$ of weight $w(e') = w(e)$. We may now find a minimum cost perfect matching in the graph $G'(X, Y, E')$ (which can be done in polynomial time) and realize that it corresponds to a minimum cost cycle cover of $G(V, E)$.