1. Theory of Linear Inequalities

For each $k = 0, \ldots, n$, let U_k denote the set of all vectors $x \in \mathbf{R}^n$ such that x has exactly k coordinates equal to $1/2$ and all other coordinates of x are either 0 or 1. Let $P \subseteq \{x \in \mathbf{R}^n : 0 \leq x \leq 1\}$ be a polyhedron and let P' denote its Chvátal closure. Show that if $U_k \subseteq P$ for some $k < n$, then $U_{k+1} \subseteq P'$.

2. Combinatorial Optimization

Let $G = (V, E)$ be a complete graph. For a vertex $v \in V$ let $\delta(v)$ denote the set of edges having v as an end; for $S \subset V$ let $\gamma(S)$ denote the set of edges having both ends in the set S; for a set $F \subset E$ let $x(F)$ denote $\sum(x_e : e \in F)$.

A Hamiltonian circuit in G is an integer solution to the following linear system:

$$
\begin{aligned}
x(\delta(v)) &= 2, \text{ for all } v \in V \\
x(\gamma(S)) &\leq |S| - 1, \text{ for all } S \neq V, |S| \geq 3 \\
0 \leq x_e &\leq 1, \text{ for all } e \in E.
\end{aligned}
$$

By a *comb* we refer to a non-empty *handle* $H \subseteq V, H \neq V$ and $2k+1$ pairwise disjoint, non-empty *teeth* $T_1, T_2, ..., T_{2k+1} \subseteq V$, for k at least 1. We require each tooth T_i to have at least one vertex in common with the handle and at least one vertex not in the handle.

Show that the comb inequality

$$x(\gamma(H)) + \sum_{i=1}^{2k+1} x(\gamma(T_i)) \leq |H| + \sum_{i=1}^{2k+1} (|T_i| - 1) - (k+1).$$

is satisfied by all Hamiltonian circuits of G by deriving the inequality as a Chvátal cut for the above linear system.

NOTE: No credit is given for an alternative proof that comb inequalities are satisfied by Hamiltonian circuits. Such a proof is given, for example, on page 988 of Schrijver's *Combinatorial Optimization*.

3. Analysis of Algorithms

We say that a 4-CNF formula is *strongly satisfiable* if it has an assignment that satisfies at least 2 literals in each clause. Design a polynomial time randomized algorithm that given a 4-CNF formula Ψ which is strongly satisfiable, finds a satisfying assignment (in the usual sense) of Ψ. The algorithm is **not** required to find a satisfying assignment if the input formula is not strongly satisfiable (even if it is satisfiable in the usual sense).

4. Graph Algorithms

Design a polynomial-time algorithm for the following problem and prove its correctness.
INPUT: A graph G and edges $e_1, e_2 \in E(G)$.
QUESTION: Do there exist disjoint cycles C_1, C_2 in G such that $e_i \in E(C_i)$?
Note. To receive full credit a complete proof is required; not just a statement of a theorem from the literature.

4. Approximation Algorithms

Consider the weighted vertex cover problem on a graph $G(V, E)$ over vertices $V = [n]$, with corresponding costs $c_i > 0$, $i \in [n]$. Show that, for any $\epsilon \in [0, 1)$, the following hold:
(a) The algorithm below is a factor $2/(1-\epsilon)$ approximation algorithm for weighted vertex cover.
(b) The analysis of (a) cannot be improved beyond factor $(2-\epsilon)/(1-\epsilon)$.
Note: You may answer the question for $\epsilon = 0$ for partial credit strictly greater than zero.

Algorithm

1. Initialization:
    - $U \leftarrow E$ (all edges are uncovered)
    - $\forall e \in E$, $y_e \leftarrow 0$
    - $C \leftarrow \emptyset$ (no vertices have been added to the vertex cover)
    - $\forall u \in V = [n]$, $\delta_u \leftarrow c_u$
2. While $U \neq \emptyset$ (thus while C is not a vertex cover) do:
    - Pick an uncovered edge $e \in U$, and let the endpoints of e be $e = (u, v)$
    - $\mu = \min(\delta_u, \delta_v)$
    - $y_e \leftarrow \mu$
    - $\delta_u \leftarrow \delta_u - \mu$
    - $\delta_v \leftarrow \delta_v - \mu$
    - Include in C all vertices having $\delta_i \leq \epsilon\, c_i$ and update U:
        - $\forall i \in V = [n]$, if $\delta_i \leq \epsilon\, c_i$ then $C \leftarrow C \cup \{i\}$
        - $U \leftarrow U \setminus \bigcup_{(i,j) \in E : i \in C} \{(i, j)\}$
3. Output C.

5. Graph Theory

Let G be a simple graph on n vertices and m edges. Prove that it has at least $\frac{m}{3n}(4m - n^2)$ triangles.

6. Probability

A **superinversion** of a permutation σ on $\{1, \ldots, n\}$ is a pair (i, j) satisfying the following two conditions:

$$\begin{aligned} (\mathbf{i}) \quad j - i &> \frac{n}{4} \\ (\mathbf{ii}) \quad \sigma(i) - \sigma(j) &> \frac{n}{4}. \end{aligned}$$

Let X_n be the number of superinversions of a permutation chosen uniformly at random from all $n!$ permutations on n elements.
(a): Compute, up to first order, $\mathbf{E}(X_n)$.
(b): Let ϵ be fixed and arbitrary. Show that

$$\mathbf{P}\left((1-\epsilon)\mathbf{E}(X_n) < X_n < (1+\epsilon)\mathbf{E}(X_n)\right) = 1 - o(1)$$

as n tends to infinity.

7. Algebra

Let R be an integral domain and suppose that $R[x]$ is a principal ideal domain. Show that R is a field.