1. Graph Theory

Given an edge-coloring of a multigraph, we say that color j is *missing* at a vertex v if no edge incident with v is colored j. Let G be a multigraph and $v_1v_2 \ldots v_n$ ($n \geq 2$) a path in G. Suppose there is an edge-coloring $c : E(G - v_1v_2) \to \{1, \ldots, k\}$ of $G - v_1v_2$, where $k \geq \Delta(G) + 1$, such that

(1) for each $2 \leq i \leq n-1$ there exists $1 \leq s \leq i-1$ such that $c(v_iv_{i+1})$ is missing at v_s, and

(2) there exist $1 \leq i < n$ and $j \in \{1, \ldots, k\}$ such that j is missing at both v_i and v_n.

Show that G is k-edge-colorable.

2. Probability

Assume that we have a Bernoulli variable X so that $P(X = 1) = 0.6$ and $P(X = 0) = 0.4$. Assume that $X, X_1, X_2, \ldots$ are i.i.d. random variables.
(a) Find the smallest $c > 0$, so that

$$P(X_1 + X_2 + \ldots + X_n \geq 0.8n) \leq c^n$$

for all n.
(b) Once you have found c, explain why c is the smallest constant satisfying the above equality for all n.

3. Analysis of Algorithms

Consider a matrix with numerical data where each entry is rational, but each row and column sum is an integer. Prove that you can "round off" this matrix, rounding each entry to the next integer above or below, without changing the row or column sums.

4. Combinatorial Optimization

Let $P = \{x : Ax \leq b\}$ be a rational polyhedron in R^n and let w and c be rational vectors such that both

$$\max \ \{w^T x : x \in P\} \tag{1}$$

and

$$\max \ \{c^T x : x \in P\} \tag{2}$$

have optimal solutions.

Give a polynomial-time procedure for finding a linear system $Bx \leq d$ that defines the set of vectors $\bar{x}$ such that $\bar{x}$ is optimal for (1) and $\bar{x}$ is optimal for (2). The description of the system $Bx \leq d$ should not involve w or c.

5. Theory of Linear Inequalities

Say that a finite set H of rational vectors in R^n is a *super Hilbert basis* if for each $J \subseteq \{e_1, \ldots, e_n\}$ the set $H \cup J$ is a Hilbert basis (where e_i denotes the ith unit vector in R^n). Let $P = \{x : Ax \leq b\}$ be a rational polyhedron in R^n. Show that $Ax \leq b, x \leq u$ is totally dual integral for every rational vector u if and only if for every face F of P the set of active rows in $Ax \leq b$ is a super Hilbert basis.

6. Algebra

Prove or disprove each of the following statements.

(a) Let p be a prime and let G be the group $(\mathbb{Z}/p\mathbb{Z})^n$. Let P be a p-Sylow subgroup of the group of autmorphisms $\mathrm{Aut}(G)$. Then, there exist sub-groups $\{e\} = P_{n-1} \subset P_{n-2} \cdots \subset P_0 = P$ such that for each $i, 1 \leq i \leq n-1$, P_i is a normal subgroup of P_{i-1} and the quotient P_{i-1}/P_i is abelian.

(b) Let G be a group, $H \subset G$ a subgroup and g an element of G such that $gHg^{-1} \subset H$. Then, $gHg^{-1} = H$.

(c) Let k be a field with $\mathrm{char}(k) = p$ and E a *finite algebraic extension* of k. Then the number of distinct intermediate fields F, $k \subset F \subset E$, is finite.

7. Randomized Algorithms

Consider the following random walk on the states $\Omega = \{0, 1, 2, \ldots, n-1\}$. From state i:

- move to state $i + 1 \bmod n$ with probability $1/2$,
- move to state 0 with probability $1/2$.

More precisely, it is a Markov chain with transition matrix P defined as follows. For $0 \leq i < n-1$:

$$P_{i,i+1} = P_{i,0} = 1/2.$$

Also, $P_{n-1,0} = 1$.

Give a coupling argument to upper bound the mixing time of the chain (within a constant factor of optimal is fine).

Recall the mixing time is defined to be

$$T = \max_{x \in \Omega} T_x$$

where

$$T_x = \min\left\{t : d_{\mathrm{TV}}(P^t(x, \cdot), \pi) \leq 1/4\right\},$$

where d_{TV} is the variation distance between the two distributions and π is the stationary distribution of the chain.

7. Approximation Algorithms

Consider the general uncapacitated facility location problem in which the connection costs are not required to satisfy the triangle inequality:

Let G be a bipartite graph with bipartition (F, C), where F is the set of *facilities* and C is the set of *cities*. Let f_i be the cost of opening facility i, and c_{ij} be the cost of connecting city j to (opened) facility i. The problem is to find a subset $I \subseteq F$ of facilities that should be opened, and a function $\phi: C \rightarrow I$ assigning cities to open facilities in such a way that the total cost of opening facilities and connecting cities to open facilities is minimized.

Give a reduction from the set cover problem to show that approximating this problem is as hard as approximating set cover and therefore cannot be done better than $O(\log n)$ factor unless $\mathbf{NP} \subseteq \widetilde{\mathbf{P}}$. Also, give an $O(\log n)$ factor algorithm for this problem.

7. Computational Complexity

Consider the FACTORING problem: Given a natural number N, express N as a product of its prime factors. To date no polynomial-time algorithm for FACTORING is known, and the conjectured hardness of FACTORING has been the basis of several public-key cryptosystems.

1. Prove that if $\mathbf{P} = \mathbf{NP} \cap \mathbf{coNP}$, then FACTORING can be solved by a polynomial-time algorithm. You may use a polynomial-time algorithm for deciding whether a given integer is prime.

2. Prove that unless $\mathbf{NP} = \mathbf{coNP}$, FACTORING is *not* **NP-Hard** under Cook reduction. Conclude that unless $\mathbf{NP} = \mathbf{PH}$, FACTORING is *not* **NP-Hard** under Cook reduction.