1. Computability, Complexity and Algorithms

Let $G = (V, E)$ be an undirected graph. Consider the following algorithm to find a large matching in G:

1. Start with $M = \emptyset$, the empty matching.
2. Add edges of G greedily to M as long as they maintain a matching.
3. If there is any edge $(u, v) \in M$ such that removing (u, v) from M allows you to add 2 new edges, then apply this change, increasing the size of M by one. Repeat this step as long as such a change is possible (augmentations of length 3).

- Show that the resulting matching M has at least $2/3$ as many edges as a maximum matching of G.
- Consider the extension where the algorithm augments on paths of length up to $2k + 1$. Show that the matching obtained has size at least $(k + 1)/(k + 2)$ times the size of the maximum cardinality matching.
- Suppose G has nonnegative weights on its edges. Show that any greedy maximal matching — choose edges in order of weight while maintaining a matching — gives a matching of at least half the weight of a maximum weight matching.

2. Analysis of Algorithms

Given an edge weighted complete bipartite graph $G = (V, E)$ and a perfect matching M in G, define $f(M)$ to be the weight of the heaviest edge in M. Define a *bottleneck perfect matching* in G to be a perfect matching N that minimizes $f(N)$.

First consider the algorithm that simply finds a minimum weight perfect matching in G. Give an example to show that the matching found by this algorithm may not be a bottleneck perfect matching. What is the approximation ratio achieved by this algorithm?

Give a polynomial time algorithm for finding a bottleneck perfect matching. Make sure your algorithm is as efficient as possible. What is its running time?

3. Theory of Linear Inequalities

Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope. Let $\mathrm{vert}(P)$ be the set of vertices of P. Let $X \subseteq \mathrm{vert}(P)$. Define $P(X) := \mathrm{conv}\,(\mathrm{vert}(P) \setminus X)$. The graph of the polytope P is a graph G_P with nodes corresponding to $\mathrm{vert}(P)$ such that two nodes are adjacent in G_P if and only if the corresponding vertices are adjacent in P (i.e. the two vertices are contained in a one-dimensional face of P).

Let $X \subseteq \text{vert}(P)$ and let $(X_1, \ldots, X_m)$ be a partition of X such that X_i and X_j are independent in G_P, i.e. there is no edge connecting X_i to X_j for all $1 \leq i < j \leq m$. Then show that

$$P(X) = \bigcap_{i=1}^{m} P(X_i).$$

4. Combinatorial Optimization

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Let $c(e)$ for $e \in E$ be the capacity of an edge. Furthermore, let $R = \{((s_1, t_1), d_1), ((s_2, t_2), d_2)\}$ be a set of two commodities, i.e., a quantity d_1 has to be send from source s_1 to sink t_1 and a quantity d_2 has to be send from source s_2 to sink t_2. Let $\delta_E(W)$ be the set of edges with exactly one endpoint in W and let $\delta_R(W)$ be the set of commodities with either its source or its sink in W but not both.

Cut condition: For each $W \subseteq V$, the capacity of $\delta_E(W)$ is not less than the demand of $\delta_R(W)$.

Euler condition:

$$\sum_{e \in \delta(v)} c(e) \equiv 0 \ (mod\ 2) \text{ if } v \neq s_1, t_1, s_2, t_2$$
$$d_1 \ (mod\ 2) \text{ if } v = s_1, t_1$$
$$d_2 \ (mod\ 2) \text{ if } v = s_2, t_2$$

We have the following theorem:

Theorem 1 *If all capacities and demands are integer and both the cut condition and the Euler condition are satisfied, then the undirected 2-commodity flow problem has an integer solution.*

Question 1. Prove the following lemma

Lemma 1 *Every cut in an Eulerian graph (with edge capacities equal to one) has even cardinality.*

Question 2. Use Theorem 1 and Lemma 1 to show the following. Let $G = (V, E)$ be an Eulerian graph and let s_1, t_1, s_2, t_2 be distinct vertices. Then the maximum number k of pairwise edge-disjoint paths $P_1, \ldots, P_k$, where each path P_j connects either s_1 and t_1 or s_2 and t_2, is equal to the minimum cardinality of a cut both separating s_1 and t_1 and separating s_2 and t_2.

5. Graph Theory

Let $k \geq 1$ be an integer and let G be a k-connected k-regular graph on an even number of vertices. Prove that G has a perfect matching.

6. Probabilistic methods

Let $X_1 \ldots, X_n$ be independent random variables with $X_i \in \{0,1\}$ and $\mathbf{Prob}[X_i = 1] = p$, for $i = 1, \ldots, n$, where $0 < p < 1$. Set $X := \sum_{i=1}^n X_i$. Prove that for any $t \in [0, 1-p]$, we have

$$\mathbf{Prob}[X \geq (p+t)n] \leq e^{-nh(p,t)},$$

where $h(p,t) = (p+t)\ln\frac{p+t}{p} + (1-p-t)\ln\frac{1-p-t}{1-p}$, and is also referred to as a "relative entropy function".

7. Algebra

(a) Suppose $K \subset H \subset G$ are groups under the same operation and that K is normal in H and H is normal in G. Does K have to be normal in G?

(b) Let G be a group and H be a subgroup of G with index n. Prove that there is a normal subgroup K of G such that $K \subset H$ and $[G:K] \leq n!$.

7. Linear Algebra

Let $T \in \mathrm{Hom}(V,V)$, where V is an n-dimensional vector space over a field $\mathbb{F}$. (In other words, T is a linear transformation from V to V.)

(i) Show that if $T^m = 0$, but $T^{m-1} \neq 0$, then there is a vector $v \in V$ such that $\{v, Tv, \ldots, T^{m-1}v\}$ is a linear independent set.

(ii) Show that if $T^m = 0$, then $T^n = 0$.

(iii) Show that if $\ker(T) \cap \mathrm{Im}(T) = \{0\}$, then $\ker(T^2) = \ker(T)$. By giving an example, show that the conclusion is false if the assumption $\ker(T) \cap \mathrm{Im}(T) = \{0\}$ does not hold.