1. Computability, Complexity and Algorithms

(a) Let G be the complete graph on n vertices, and let $c : V(G) \times V(G) \to [0, \infty)$ be a symmetric cost function. Consider the following *closest point heuristic* for building a low cost traveling salesman tour. Begin with a trivial tour consisting of a single arbitrarily chosen vertex. At each step, identify the vertex u that is not on the tour but whose distance to a vertex on the tour is minimum. Suppose that the vertex on the tour that is nearest to u is v. That is, $c(u, v) = \min\{c(u', v') : u' \text{ is not in the tour}, v' \text{ is in the tour}\}$. Extend the tour to include u by inserting u just after v. More precisely, if the tour consists of the single vertex v, then replace it by the tour vuv. Otherwise pick an edge vw of the tour, remove it from the tour and add the edges vu and uw instead. Repeat until all vertices are on the tour. Prove that, if the cost function satisfies the triangle inequality, then the closest point heuristic returns a tour whose total cost is not more than twice the cost of an optimal tour.
(b) Show how, in polynomial time, we can transform one instance of the general traveling salesman problem (i.e. when the cost function is not guaranteed to satisfy triangle inequality) into another instance whose cost function satisfies triangle inequality and such that the two instances have the same optimal tours. Can such a polynomial time transformation be used to obtain a factor 2 approximation algorithm for the general traveling salesman problem? Justify your answer.

2. Analysis of Algorithms

(a) Recall that most known problems in the class NP exhibit the property of self-reducibility. Given an oracle for the decision version of the problem, this property helps yield a polynomial time algorithm for finding a solution to a "YES" instance of the problem. Give a proof of self-reducibility for the problem CLIQUE: Given an graph G and a number k, find a clique of size k in G.
(b) Recall the Isolating Lemma:
Let S be a set, $|S| = n$, and F be a family of subsets of S. Assign random weights to elements of S from $\{1, 2, \ldots, 2n\}$. Then with probability at least $1/2$, there is a unique minimum weight set in F.

In this lemma, the weight of a set $A \in F$ is defined to be the sum of weights of elements in A. Prove the same lemma if the weight of a set $A \in F$ is defined to be the *product* of weights of elements in A.

3. Theory of Linear Inequalities

For a matrix A having m rows and a set $S \subseteq \{1, ..., m\}$, let A_S denote the submatrix of A consisting of the rows indexed by S. Let $\mathbf{1}$ denote the vector consisting of all 1's.

Let A be an $m \times n$ integral matrix and let b be a rational vector such that the linear system $Ax \leq b$ has at least one solution. Show that $Ax \leq b$ is totally dual integral if and only if (1) the rows of A form a Hilbert basis and (2) for each subset S of at most n inequalities from $Ax \leq b$, the linear programming problem $\min\{y^T b : y^T A = \mathbf{1}^T A_S,\ y \geq 0\}$ has an integral optimal solution.

4. Combinatorial Optimization

Let $\mathcal{S}$ be a finite set, let $X_1, \ldots, X_t$ be a partition of $\mathcal{S}$, and let $Y_1, \ldots, Y_l$ be a partition of $\mathcal{S}$. Consider the polytope P defined by $x \in \mathbb{R}^{\mathcal{S}}$ such that

$$\begin{aligned} &x(u) \geq 0 \text{ for all } u \in \mathcal{S},\\ &x(X_i) \leq 1 \text{ for all } 1 \leq i \leq t,\\ &x(Y_i) \leq 1 \text{ for all } 1 \leq i \leq l. \end{aligned}$$

Show that P is integral.

5. Graph Theory

Let $k \geq 1$ be an integer, let G be a 2-connected graph, let x, y be distinct vertices of G, and assume that every vertex of G other than x or y has degree at least k. Prove that G has a path with ends x and y of length at least k.

6. Probabilistic methods

Let $v_1, v_2, \ldots v_n$ be n vectors from $\{\pm 1\}^n$ chosen uniformly and independently. Let M_n be the largest pairwise dot product in absolute value: i.e

$$M_n = \max_{i \neq j} |v_i \cdot v_j|$$

Prove that

$$\frac{M_n}{2\sqrt{n \ln n}} \to 1$$

in probability as $n \to \infty$.
Hint. Consider the first and second moment methods applied to the number of pairs of vectors whose dot product exceeds (and falls below, respectively) $2\sqrt{n \ln n}$.

7. Algebra

Let F be a finite field with cardinality q.

(i) For how many $a \in F$ does the polynomial $x^5 - a$ have a root in F?

(ii) For how many $a \in F$ does the polynomial $x^5 - a$ split completely into linear factors over F?

Express your answers in terms of q.

7. Linear Algebra

Given $A \in \mathbb{R}^{m\times n}$ with $m > n$ and $\underline{j} = (j_1, \ldots, j_n) \in [1, m]^n$, call $A^{\underline{j}}$ the $n \times n$ minor of A formed by the j_i-th, $i = 1, \ldots, n$, rows of A. Given $A, B \in \mathbb{R}^{m\times n}$ show that

$$\det(A^T B) = \sum_{\underline{j}\in J} \det(A^{\underline{j}}) \det(B^{\underline{j}}),$$

where J is the set of multi-indexes $\underline{j}$ such that $1 \leq j_1 < j_2 < \ldots < j_n \leq m$. Use the above to show that

$$\det(A^T B)^2 \leq \det(A^T A) \det(B^T B).$$