1. Analysis of Algorithms

Consider the following two graph problems:

Graph coloring: Given a graph $G = (V, E)$ and an integer $c \geq 0$, a c-coloring is a function $f : V \to \{1, 2, \ldots, c\}$ such that $f(u) \neq f(v)$ for all edges $uv \in E(G)$. We define $\chi(G)$ to be the minimum integer $c \geq 0$ such that G has a c-coloring. An α-approximation algorithm for the coloring problem is one that, given a graph G, returns a c-coloring, where $c \leq \alpha\chi(G)$.

Longest path: Given a graph $G = (V, E)$, a *path* P in G is a sequence of distinct vertices $v_1, \ldots, v_d$ such that $v_i v_{i+1} \in E$ for all $1 \leq i < d$, and the length of the path P is d. We wish to find a path which maximizes d. We write that the longest path uses $\ell(G)$ vertices. An α-approximation algorithm for the longest path problem is one that, given a graph G, returns a path of length at least $\ell(G)/\alpha$.

Unfortunately, unless P = NP, there is no polynomial time $n^{1-\epsilon}$-approximation algorithm for either problem, for any constant $\epsilon \in (0, 1)$, where $n = |V|$. Taken together, however, the problems are easier. In this question, you will give a linear time algorithm that, for any graph G and for any $0 < \epsilon < 1$, outputs either an $n^{1-\epsilon}$-approximation to the graph coloring problem, or a n^{ϵ}-approximation to the longest path problem.

(a) For any c-coloring f of a graph $G = (V, E)$, the color class of color i is defined to be $C(i) = \{v \in V : f(v) = i\}$. Given an n vertex graph, use DFS to find some parameter k such that there is a k-coloring as well as a path in the graph of length k so that each color class of the coloring contains exactly one vertex on the path.

(b) Using part (a), for any graph G and parameter $\epsilon \in (0, 1)$, show how to output in linear time either a path that is an n^{ϵ}-approximation for the longest path problem, or a c-coloring that is an $n^{1-\epsilon}$-approximation for the graph coloring problem. (Hint: Observe that each path has length at most n and each coloring has at least 1 color.)

2. Approximation Algorithms

Minimizing Shipping Times. A factory produces m different kinds of items (say $\{1, \ldots, m\}$), and produces one item of each kind every time step. Let $\mathcal{R} = \{R_1, \ldots R_n\}$ be a set of customer requests. Every request contains at most one item of each kind, and hence a request R_i is specified by a subset $R_i \subseteq \{1, \ldots, m\}$.

The items produced by the factory are to be allocated to the requests $\mathcal{R}$. The shipping time of a request R_i is the time at which all its items are allocated. The goal is to find an allocation that minimizes the total shipping time of all the requests.

In particular, design a polynomial time algorithm that outputs an allocation whose total shipping time is within a factor 2 of the optimal allocation (a 2-approximation algorithm).

(Hint: Write a linear program with **only** the shipping times of the requests as variables.)

3. Theory of Linear Inequalities

Let P be a non-empty bounded polyhedron in R^n, let $c \in R^n$, and let $a_i \in R^n$ and $b_i \in R$ for $i = 1, \ldots, t$ for some positive integer t. Show that the linear programing problem

$$\max(c^T x : x \in P, a_1^T x = b_1, \ldots, a_t^T x = b_t)$$

has an optimal solution that is a convex combination of at most $t + 1$ vertices of P.

4. Combinatorial Optimization

Let $G = (V, E)$ be a complete graph, with vertex set V and edge set E; let $n = |V|$. Consider the polytope P defined by

$$x_i + x_j \leq 1, \text{ for all } \{i, j\} \in E$$
$$0 \leq x_i \leq 1, \text{ for all } i \in V.$$

Show that the Chvátal rank of P is at least $\log_2(n - 1)$.

5. Graph Theory

Let G be a connected graph. (a) Use a depth-first-search spanning tree to prove that if G is triangle-free, then G contains a bipartite subgraph H such that $|E(H)| \geq 3(|V(G)| - 1)/4$ and every component of H is an induced subgraph of G. (b) Prove an analogous bound when G has odd girth $g \geq 5$ by replacing the constant $3/4$ by an appropriate function of g. [A graph G has *odd girth* g if g is the largest integer such that every odd cycle in G has length at least g.]

6. Probability

Let $(X_n)_{n=1}^{\infty}$ be a sequence of i.i.d. random variables taking values ± 1 with probability $1/2$. Let

$$S_n = \begin{cases} X_1 + \ldots + X_n, & n = 1, 2, \ldots \\ 0, & n = 0. \end{cases}$$

Let $S_n^* = \max_{k=1,\ldots,n} S_k$, $n = 1, 2, \ldots$

Find the density of the limiting distribution for $\frac{S_n^*}{\sqrt{n}}$ and prove the convergence using the following "reflection principle": for any $n, r = 1, 2, \ldots$,

$$\mathsf{P}\{S_n^* > r\} = 2\mathsf{P}\{S_n > r + 1\} + \mathsf{P}\{S_n = r + 1\}.$$

You do not need to prove the reflection principle.

7. Algebra

Let $G = GL(2, \mathbb{C})$ denote the group of invertible 2 by 2 matrices with complex entries, and B denote the subgroup of G that consists of all matrices with lower-left entry 0. Recall that the double-coset $[g]$ of $g \in G$ with respect to B is the set $\{bgb' \,| b, b' \in B\}$. Let $X = \{[g] \,| g \in G\}$ denote the set of double cosets of B in G. Prove that there exist two matrices g_0 and g_1 in G such that $X = \{[g_0], [g_1]\}$.