ACO Comprehensive Exam Spring 2025

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1 Algorithms

We are given three arrays $A[0...n-1]$, $B[0...m-1]$, $M[0...m-1]$ with $m \leq n$ and $M[i] \in [0,1]$ for all i. For vectors $v \in \mathbb{R}^m$ we define the *mask-norm*

$$\|v\|_M^2 := \sum_{k=0}^{m-1} M[k] \cdot v_k^2$$

i.e., the classical L2-norm but we weight entries by $M[k] \in [0,1]$. So array M functions as a "mask" and each entry $M[k]$ specifies how important the entry is on a scale from 0 to 1. For any index i, we interpret $A[i...i+m-1], B[0...m-1] \in \mathbb{R}^m$ as m-dimensional vectors. We want to find index i where $A[i...i+m-1]$ is "closest" to $B[0...m-1]$ with respect to the mask-norm. That is, find i that minimizes

$$\|A[i...i+m-1] - B[0...m-1]\|_M^2 = \sum_{k=0}^{m-1} M[k] \cdot (A[i+k] - B[k])^2.$$

(For example, array B may be a picture of some object, array M is a mask that separates the object from the background in picture B, and we try to find the object within picture A.)

1. (2 points) Show that for any two vectors $u, v \in \mathbb{R}^m$, we have

$$\|u - v\|_M^2 = \|u\|_M^2 - 2\left(\sum_{k=0}^{m-1} u_k \cdot v_k \cdot M[k]\right) + \|v\|_M^2.$$

2. (4 points) Design and analyze an algorithm that, given arrays $C[0...n-1]$, $D[0...m-1]$, computes in $O(n \log n)$ time the array $E[0...n-m]$ with

$$E[i] = \sum_{k=0}^{m-1} C[i+k] \cdot D[k]$$

(Hint: Use FFT in $O(n \log n)$ time for polynomial multiplication.)

3. (4 points) Design and analyze an algorithm that, given arrays $A[0...n-1]$, $B[0...m-1]$, $M[0...m-1]$, returns in $O(n \log n)$ time an index i where $\|A[i...i+m-1] - B[0...m-1]\|_M^2$ is as small as possible. (You may use 2. even if you have not solved it.)

2 Graph Theory

Question. Show that for every positive integer k there exists a positive integer c_k with the following property: If G is a c_k-vertex-connected graph for which there exists $S \subseteq V(G)$ with $|S| = k$ such that

(1) $G - S$ is a bipartite graph with partition sets V_1, V_2, and

(2) for every $T \subseteq S$ and for each $i \in [2]$, $|N_G(T) \cap V_i| \geq |T|$,

then G contains k odd cycles that are pairwise vertex disjoint. Give a sufficient bound on c_k as a function of k alone.

3 Linear Inequalities

A sequence of positive integers $\{a_j\}_{j\in[n]}$ is called super-increasing if $\sum_{j=1}^{k} a_j < a_{k+1}$ for all $k \in [n-1]$. For example, the sequence $a_j = 2^{j-1}$ for $j \in [n]$ is super-increasing. Consider a knapsack set with coefficients being super-increasing:

$$S := \left\{ x \in \{0,1\}^n \;\middle|\; \sum_{j=1}^{n} a_j x_j \le b \right\},$$

where we assume $a_n \le b$ and there exists $\mathcal{I} \subseteq [n]$ such that $\sum_{j\in\mathcal{I}} a_j = b$. For $j \in [n] \setminus \mathcal{I}$, define the set: $\mathcal{I}_j := \{j\} \cup \{i \in \mathcal{I} \,|\, i > j\}$.

(a.) (3 pt) Show that if $\hat{x} \in S$, then it satisfies the inequalities:

$$\sum_{i\in\mathcal{I}_j} \hat{x}_i \le |\mathcal{I}_j| - 1, \tag{1}$$

for all $j \in [n] \setminus \mathcal{I}$.

(b.) (3 pt) Show that if $\hat{x} \in \{0,1\}^n \setminus S$, then there exists $j' \in [n] \setminus \mathcal{I}$ such that $\hat{x}$ does not satisfy (1) corresponding to j'.

(c.) (4 pt) Show that $P := \{x \in [0,1]^n \,|\, (1),\ \forall j \in [n] \setminus \mathcal{I}\}$ is an integral polytope, thus establishing that $\mathrm{conv}(S) = P$.