ACO Comprehensive Exam Fall 2025

Aug 14, 2025

1 Design and Analysis of Algorithms

Let $G = (V, E)$ be a connected, undirected graph. The goal of this problem is to design an "optimal" random walk to sample uniformly from the vertices of G. The random walk is only allowed to use the edges of the graph and the probability of using an edge in either direction must be equal (the walk is "symmetric").

Let $X \in \mathbb{R}^{|V|\times|V|}$ denote the weights assigned to the edges. Then we have

$$X \geq 0, \quad X_{ij} = 0 \ \forall ij \notin E, \quad X = X^\top, \quad X\mathbf{1} = \mathbf{1}. \tag{1}$$

1. (1 pt) Show that for any matrix satisfying (1), for the corresponding random walk on G, the uniform distribution on the vertices is stationary.
2. (2 pts) Show that the largest eigenvalue of any such X is 1.
3. (3 pts) Write down a convex program whose feasible region is the set of matrices satisfying (1) and whose objective is to minimize the second largest eigenvalue of X (second largest in absolute value). Prove that your program is convex. [You are not required to bound the second largest eigenvalue.]
4. (2 pts) Write down an equivalent SDP.
5. (2 pts) What is the complexity of solving the above convex programs using an efficient cutting plane method? State both the number of separation oracle queries and the time to solve each query.

2 Combinatorial Optimization

Let $G = (S, T, E)$ be a bipartite graph with bipartition S and T. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid with rank function $r : 2^S \to \mathbb{Z}$.

1. (2 pts) Show that $\mathcal{M}' = (E, \mathcal{I}')$ is a matroid where $\mathcal{I}' = \{F \subseteq E : d_F(v) \leqslant 1 \ \forall v \in S$ and the subset of S covered by F is in $\mathcal{I}$ $\}$.

2. Consider the following functions $b : 2^E \to \mathbb{Z}$ and $p : 2^E \to \mathbb{Z}$ defined as follows. For every $F \subseteq E$, we have $b(F) = r(\{u \in S : \exists \{u, v\} \in F\})$ as the rank of the subset of S covered by F. For every $F \subseteq E$, let $p(F) = |\{v \in T : \delta(v) \subseteq F\}|$ be the number of nodes v in T for which every edge of G incident at v is in F.

   (a) (2 pts) Show that b is a submodular function and p is a supermodular function.

   (b) (3 pts) Suppose that for every $X \subseteq T$, we have the following generalized Hall's condition
   $$r(\Gamma_S(X)) \geqslant |X|,$$
   where $\Gamma_S(X)$ denotes the set of neighbors of X in S. Show that $b(F) \geqslant p(F)$ for each $F \subseteq E$.

3. (3 pts) Under the assumption in the previous part, apply the integer-valued version of the Discrete Sandwich Theorem to p and b and show that it implies that G has a matching M covering T such that $M \in \mathcal{I}'$.

3 Probabilistic Combinatorics

Let $p(n), q(n), r(n)$ be three functions $\mathbb{N} \to [0,1]$ satisfying $p(n) + q(n) + r(n) = 1$ and $p(n) \geq q(n) \geq r(n) \geq n^{-1.9}$. Let χ be a random 3-coloring of the edges of K_n where each edge is independently colored red with probability $p(n)$, blue with probability $q(n)$, and green with probability $r(n)$. A *rainbow triangle* in χ is a triangle with one edge of each color.

Find (with proof) a function $t(n)$ such that:

(a) (3 pts) If $p(n)q(n)r(n) = o(t(n))$, then w.h.p. χ has no rainbow triangles.

(b) (5 pts) If $p(n)q(n)r(n) = \omega(t(n))$, then w.h.p. χ has at least one rainbow triangle.

(c) (2 pts) Finally, for your choice of $t(n)$, exhibit functions $p(n), q(n), r(n)$ satisfying $p(n) + q(n) + r(n) = 1$ and $p(n)q(n)r(n) = \omega(t(n))$, but NOT $p(n) \geq q(n) \geq r(n) \geq n^{-1.9}$, for which w.h.p. χ has no rainbow triangles.