ACO Comprehensive Exam Spring 2024

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1 Algorithms

Problem (MAX-SAT with all positive variables)

Consider a Boolean function f in Conjunctive Normal Form with n Boolean variables $x_1, x_2, \dots x_n$ and m clauses $C_1, C_2, \dots, C_m$ such that all variables appear positively in all clauses. This problem has a trivial satifying assignment (all variables set to True).

(a) (4 points) You are given nonnegative weights $w_1, w_2, \dots, w_n$, one for each variable, and $w'_1, w'_2, \dots, w'_m$, one for each clause. Your goal is to maximize the sum of the weights of satisfied clauses plus the sum of the weights of the variables set to false. Note that you do not need to get a satisfying assignment! Write an `Integer Linear Program` for this problem with one variable y_i for each Boolean variable x_i and one variable z_j for each clause C_j. Prove the equivalence of your proposed ILP with the original problem.

(b) (2 points) Set each variable to True with probability y_i^*, where $\{y_i^*\}_{i=1}^n$ is the solution of the LP relaxation of the Integer Program from part (a). Bound the ratio of the expected weight of the solution obtained by this algorithm to the weight of the optimal solution.

(c) (4 points) Now set each variable to True with probability $1 - \lambda + \lambda y_i^*$, where λ is a scalar to be set later and $\{y_i^*\}_{i=1}^n$ is the solution of the LP relaxation from part (a). Bound the ratio of the expected weight of the solution obtained by this algorithm to the weight of the optimal solution, as an expression in terms of λ. Choose the value of λ in the algorithm to obtain a better approximation ration than in part (b).

Solution

(a) We set binary variables y_i equal to 1 if and only if x_i is set to be True and binary variables z_j to be 1 if and only if clause C_j evaluates to True. Our ILP is

$$\max\ W = \sum w_i(1 - y_i) + \sum w'_j z_j$$

with the following restrictions:

$$\sum_{x_i \in C_j} y_i \geq z_j \text{ for all } 1 \leq j \leq m.$$

The restrictions guarantee that $z_j = 0$ when all $x_i \in C_j$ satisfies $x_i = 0$. With this equivalence, it follows that the function W corresponds to the weight of true clauses plus the weights of those variables set to false, and any clause C_j is set to true if and only if there is at least one literal on it set to true.

(b) Let $(\hat{y}, \hat{z})$ be the randomized solution. Since $\hat{z}_j = 0$ if and only if $\sum_{i:x_i \in C_j} \hat{y}_i = 0$, we have

$$\mathbb{P}(\hat{z}_j = 0) = \prod_{i:x_i \in C_j} \mathbb{P}(y_i = 0) = \prod_{i:x_i \in C_j} (1 - y_i^*).$$

Set $|C_j| = k$. Using the Arithmetic Mean-Geometric Mean inequality we get

$$\mathbb{P}(\hat{z}_j = 0) \leq \left(1 - \frac{\sum y_i^*}{k}\right)^k \leq \left(1 - \frac{z_j^*}{k}\right)^k$$

where the last inequality uses the constraints of the `ILP` instance from part (a). Denoting by $\hat{W}$ the weight of the solution obtained by the randomized algorithm, we compute

$$\mathbb{E}(\hat{W}) \geq \sum w_i(1 - y_i^*) + \sum w_j' \left(1 - \left(1 - \frac{z_j^*}{k}\right)^k\right) \tag{1}$$

Let $f(z) = 1 - \left(1 - \frac{z}{k}\right)^k$. One can check that $f'(z) > 0, f''(z) < 0$ and so

$$f(z) \geq f(1)z = \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z \geq \left(1 - \frac{1}{e}\right) z$$

holds for all $z \in [0, 1]$ (the curve from $(0, f(0))$ to $(1, (f(1))$ stays above the straight line between these points). Substituting back in (1) we conclude

$$\begin{aligned}
\mathbb{E}(\hat{W}) &\geq \sum w_i(1 - y_i^*) + \sum w_j' \left(1 - \frac{1}{e}\right) z_j^* \\
&\geq \left(1 - \frac{1}{e}\right) \left(\sum w_i(1 - y_i^*) + \sum w_j' z_j^*\right) \\
&\geq \left(1 - \frac{1}{e}\right) OPT_{ILP}
\end{aligned}$$

where OPT_{ILP} is the weight of the optimal solution. In the last inequality, we use that the optimal value of the LP relaxation is always greater or equal to the optimal value of the `ILP`.

(c) As in part (b), we compute the probability of the events $\{\hat{z}_j = 0\}$, this time as a function of λ. Analogous calculations yield

$$\mathbb{P}(\hat{z}_j = 0) = \prod_{i:x_i \in C_j} \mathbb{P}(x_i = 0) = \lambda^{|C_j|} \prod_{i:x_i \in C_j} (1 - y_i^*).$$

Using again the Arithmetic Mean-Geometric Mean inequality and setting $|C_j| = k$, we get

$$\mathbb{P}(C_j = 0) \leq \lambda^k \left(1 - \frac{\sum y_i^*}{k}\right)^k \leq \lambda^k \left(1 - \frac{z_j^*}{k}\right)^k$$

Let $f_\lambda(z) = 1 - \lambda^k \left(1 - \frac{z}{k}\right)^k$. One can check that $f_\lambda''(z) < 0$ for $\lambda > 0$, and then

$$f_\lambda(z) \geq f_\lambda(1) z$$

for all $z \in [0, 1]$. Finally, we see that $f(1) = 1 - \lambda^k \left(1 - \frac{1}{k}\right)^k \geq 1 - \lambda^2/4$ for all $k \geq 2$. Clauses C_j with one literal can be analysed separately to check they satisfy $\mathbb{P}(\hat{z}_j = 1) \geq \lambda z_j^*$ as long as $\lambda < 1$. Combining all these inequalities, we get the following lower bound on $\hat{W}_\lambda$, the weight of this randomized algorithm:

$$\begin{aligned}
\mathbb{E}(\hat{W}_\lambda) &\geq \sum w_i \lambda (1 - y_i^*) + \sum w_j' \left(1 - \lambda^k \left(1 - \frac{z_j^*}{k}\right)^k\right) \\
&\geq \lambda \sum w_i (1 - y_i^*) + \min(\lambda, 1 - \lambda^2/4) \sum w_j' z_j^* \\
&\geq \min(\lambda, 1 - \lambda^2/4) \left(\sum w_i (1 - y_i^*) + \sum w_j' z_j^*\right) \\
&\geq \min(\lambda, 1 - \lambda^2/4) OPT_{ILP}
\end{aligned}$$

The optimal parameter arises when $\lambda = 1 - \lambda^2/4$ which yields a $2(\sqrt{2} - 1)$ lower bound. Note that $2(\sqrt{2} - 1) \approx 0.828 > 0.632 \approx (1 - e^{-1})$.

2 Graph Theory

Question. Let G be a 2-connected graph and $x_1, x_2 \in V(G)$ be distinct, and let n_1, n_2 be positive integers such that $n_1+n_2 = |V(G)|$. Show that G contains vertex disjoint subgraphs G_1 and G_2 such that, for $i \in [2]$, G_i is connected, $x_i \in V(G_i)$, and $|V(G_i)| = n_i$. (Hint: Use ear decomposition, a decomposition into a cycle and paths.)

Solution. We show that there is a linear ordering of the vertices of G, $v_1 < v_2 < \ldots < v_n$, such that $v_1 = x_1$, $v_n = x_2$, and for any $i \in [n] \setminus \{n\}$, both $G[\{v_1, \ldots, v_i\}]$ and $G[\{v_{i+1}, \ldots, v_n\}]$ are connected. Then the statement follows by letting $i = n_1$.
Let $G + x_1x_2 = G$ if $x_1x_2 \in E(G)$; otherwise let $G + x_1x_2$ be obtained from G by adding an edge beween x and x_2. Then $G + x_1x_2$ is 2-connected, and the edge x_1x_2 is contained in a cycle in $G + x_1x_2$. Whitney's theorem on ear decomposition states that $G + x_1x_2$ has a decomposition $P_0, P_1, \ldots, P_k$ such that

- P_0 is a cycle and $x_1x_2 \in E(P_0)$,
- for each $i \in [k]$, P_i is a $\left(\bigcup_{j=0}^{i-1} P_j\right)$-path, and
- $\bigcup_{j=0}^{k} P_j = G$.

We apply induction on k to prove the above assertion on linear ordering.
For the base case, let $k = 0$. Then $G + x_1x_2 = P_0$ is a cycle and we may order the vertices along the path $P_0 - x_1x_2$ from x_1 to x_2 as $v_1 < v_2 < \ldots < v_n$ with $v_1 = x_1$ and $v_n = x_2$. Note that, for each $i \in [n] \setminus \{n\}$, both $G[\{v_1, \ldots, v_i\}]$ and $G[\{v_{i+1}, \ldots, v_n\}]$ are paths and, hence, must be connected.
Now let $t \geq 1$ be an integer such that the assertion is true when $k < t$, and consider the case when $k = t$. Let H be obtained from G by removing all edges and internal vertices of P_t. Then H is 2-connected, $P_0, P_1, \ldots, P_{t-1}$ is an ear decomposition of H, and $xy \in E(P_0)$. So by induction hypothesis, we may linearly order the vertices of H as $u_1 < u_2 < \ldots < u_m$ such that $u_1 = x_1$ and $u_m = x_2$ and for each $i \in [m] \setminus \{m\}$, both $H[\{u_1, \ldots, u_i\}]$ and $H[\{u_{i+1}, \ldots, u_m\}]$ are connected.
Let $P_t = w_1 \ldots w_s$. Then $w_1 = u_p$ and $w_s = u_q$ for some $p, q \in [m]$, and we may assume $p < q$. Order the vertices of G as $u_1 < \ldots < u_p < w_2 < \ldots < w_{s-1} < u_{p+1} < \ldots < u_m$ and relabel them as $v_1 < v_2 < \ldots < v_n$. We claim that this gives the desired linear ordering. To see this, fix $i \in [n] \setminus \{n\}$.

Case 1. $v_i = u_i$ for some $i \leq p$.
Then $G[\{v_1, \ldots, v_i\}] = H[\{u_1, \ldots, u_i\}]$ is connected. Note that $H[\{u_{i+1}, \ldots, u_m\}]$ and the path $P_t - u_i$ span $G[\{v_{i+1}, \ldots, v_n\}]$. Hence, $G[\{v_{i+1}, \ldots, v_n\}]$ must be connected.

Case 2. $v_i = w_r$ for some r with $2 \leq r \leq s-1$. (This case occurs only when $s \geq 3$.)
In this case, $H[\{u_1, \ldots, u_p\}]$ and the path $w_1 \ldots w_r$ span $G[\{v_1, \ldots, v_i\}]$ which, therefore, must be connected. Also, $H[\{u_{p+1}, \ldots, u_m\}]$ and the path $w_{r+1} \ldots w_s$ span $G[\{v_{i+1}, \ldots, v_n\}]$;

so $G[\{v_{i+1}, \ldots, v_n\}]$ is connected.

Case 3. $v_i = u_l$ for some l with $p + 1 \leq l \leq m$.
Then $G[\{v_{i+1}, \ldots, v_n\}] = H[\{u_{l+1}, \ldots, u_m\}]$ is connected. Moreover, $H[\{u_1, \ldots, u_l\}]$ and the path $P_t - u_l$ span $G[\{v_1, \ldots, v_i\}]$; so $G[\{v_1, \ldots, v_i\}]$ must be connected.

3 Linear Inequalities

1. (2 pt) Give an example of a polytope in dimension n defined by exactly $2n$ inequalities, such that removing any inequality describing the polytope makes the resulting polyhedron unbounded.
   Solution. Consider the unit cube, $P = \{x \in \mathbb{R}^n | 0 \leq x_j \leq 1 \forall j \in \{1, \dots, n\}\}$, that is it a polytope described using $2n$ inequalities. If we remove any inequality, clearly the resulting polyhedron is unbounded.

2. Consider a polyhedron $P = \{x \in \mathbb{R}^n | Ax \leq b\} \neq \emptyset$ where $A \in \mathbb{R}^{m \times n}$.

   (a) (3 pt) Show that $\{c \in \mathbb{R}^n | \exists y \geq 0, A^\top y = c\} = \mathbb{R}^n$ iff P is bounded.
   Solution. Consider the primal-dual pair:

   $$\begin{array}{rl} \max & c^\top x \\ \text{s.t.} & Ax \leq b. \end{array} \qquad \begin{array}{rl} \min & b^\top y \\ \text{s.t.} & A^\top y = c \\ & y \geq 0. \end{array}$$

   $\Rightarrow$ Given a $c \in \mathbb{R}^n$, since the dual is feasible, it either has an optimal solution or is unbounded. However, the dual cannot be unbounded, since $P \neq \emptyset$. Thus, the dual has an optimal solution for all $c \in \mathbb{R}^n$. By strong duality, the primal has an optimal solution for all $c \in \mathbb{R}^n$. Thus, P is bounded.
   $\Leftarrow$ Since $P \neq \emptyset$ and if P is bounded, therefore the primal has an optimal solution for all $c \in \mathbb{R}^n$. Thus, the dual is feasible for all $c \in \mathbb{R}^n$, i.e., $\{c \in \mathbb{R}^n | \exists y \geq 0, A^\top y = c\} = \mathbb{R}^n$.

   (b) (1 pt) Suppose rank$(A) = n$ and let $A^\top = [a_1, \dots, a_n, \dots, a_m]$ where we assume (WLOG) that the matrix $[a_1, \dots, a_n]$ is non-singular (i.e., the left-hand-side of the first n constraints of $Ax \leq b$ are linearly independent). Let $B = ([a_1, \dots a_n])^{-1}$. Show that $\{c \in \mathbb{R}^n | \exists y \geq 0, BA^\top y = c\} = \mathbb{R}^n$ if and only if P is bounded.
   Solution. This follows from part (a) above and the fact that $\{c \in \mathbb{R}^n | \exists y \geq 0, BA^\top y = c\} = \mathbb{R}^n$ iff $\{c \in \mathbb{R}^n | \exists y \geq 0, A^\top y = B^{-1}c\} = \mathbb{R}^n$ iff $\{c \in \mathbb{R}^n | \exists y \geq 0, A^\top y = c\} = \mathbb{R}^n$, since B is non-singular.

   (c) (1 pt) Suppose rank$(A) = n$ and B is as defined above. Let $\{e^1, e^2, \dots, e^n\}$ be the n-standard unit vectors and $-\mathbf{1} = [-1, -1, \dots, -1]^\top$. Show that if P is bounded, then all these vectors lie in the conic combinations of a subset of columns of $BA^\top$ where the cardinality of this subset is at most $2n$.
   Solution. Let $BA^\top = [e^1, e^2, \dots, e^n, \tilde{a}_{n+1} \dots, \tilde{a}_m]$. Clearly, $\{e^1, e^2, \dots, e^n\}$ are conic combinations of the first n columns of $BA^\top$. Now consider $-\mathbf{1}$. We know from part(b) that $-\mathbf{1}$ is a conic combination of columns of $BA^\top$. Note that $-\mathbf{1}$ can be written as a conic combination of n columns (e.g., simplex standard form argument) of $BA^\top$. Therefore,$\{e^1, e^2, \dots, e^n\}$ and $-\mathbf{1}$ are in conic combination of at most $2n$ columns of $BA^\top$.

   (d) (3 pt) Show that if P is bounded and $m > 2n$, then one can always select a constraint in the system $Ax \leq b$ such that removing this inequality leaves the resulting polyhedron bounded.

Solution. Since P is bounded, this implies that the lineality space of P is $\{0\}$. Since the lineality space of P is equal to $\{x|Ax = 0\}$, we have that the rank(A) is the number of columns of A, which is n.

Therefore, by part(c) $(e^1, e^2, \ldots, e^n)$ and $-\mathbf{1}$ can all be written as conic combination of at most $2n$ columns of $BA^\top$. Thus, any vector in $\mathbb{R}^n$ can be written as a conic combination of these $2n$ columns of $BA^\top$. WLOG, let these be the first $2n$ columns of $BA^\top$. Since $m > 2n$, we can therefore drop a constraint $a_j^\top x \leq b$ with $j > 2n$. The resulting left-hand-side matrix still has rank n. Thus, by part (b), this polyhedron is bounded.