ACO Comprehensive Exam Spring 2024

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1 Algorithms

Problem (MAX-SAT with all positive variables)

Consider a Boolean function f in Conjunctive Normal Form with n Boolean variables $x_1, x_2, \dots x_n$ and m clauses $C_1, C_2, \dots, C_m$ such that all variables appear positively in all clauses. This problem has a trivial satifying assignment (all variables set to True).

(a) (4 points) You are given nonnegative weights $w_1, w_2, \dots, w_n$, one for each variable, and $w'_1, w'_2, \dots, w'_m$, one for each clause. Your goal is to maximize the sum of the weights of satisfied clauses plus the sum of the weights of the variables set to false. Note that you do not need to get a satisfying assignment! Write an `Integer Linear Program` for this problem with one variable y_i for each Boolean variable x_i and one variable z_j for each clause C_j. Prove the equivalence of your proposed ILP with the original problem.

(b) (2 points) Set each variable to True with probability y_i^*, where $\{y_i^*\}_{i=1}^n$ is the solution of the LP relaxation of the Integer Program from part (a). Bound the ratio of the expected weight of the solution obtained by this algorithm to the weight of the optimal solution.

(c) (4 points) Now set each variable to True with probability $1 - \lambda + \lambda y_i^*$, where λ is a scalar to be set later and $\{y_i^*\}_{i=1}^n$ is the solution of the LP relaxation from part (a). Bound the ratio of the expected weight of the solution obtained by this algorithm to the weight of the optimal solution, as an expression in terms of λ. Choose the value of λ in the algorithm to obtain a better approximation ration than in part (b).

2 Graph Theory

Let G be a 2-connected graph and $x_1, x_2 \in V(G)$ be distinct, and let n_1, n_2 be positive integers such that $n_1 + n_2 = |V(G)|$. Show that G contains vertex disjoint subgraphs G_1 and G_2 such that, for $i \in [2]$, G_i is connected, $x_i \in V(G_i)$, and $|V(G_i)| = n_i$. (Hint: Use ear decomposition, a decomposition into a cycle and paths.)

3 Linear Inequalities

(i) (2pt) Give an example of a polytope in dimension n defined by exactly $2n$ inequalities, such that removing any inequality describing the polytope makes the resulting polyhedron unbounded.

(ii) Consider a polyhedron $P = \{x \in \mathbb{R}^n | Ax \leq b\} \neq \emptyset$ where $A \in \mathbb{R}^{m \times n}$.

(a) (3pt) Show that $\{c \in \mathbb{R}^n | \exists y \geq 0, A^\top y = c\} = \mathbb{R}^n$ iff P is bounded.

(b) (1pt) Suppose $\text{rank}(A) = n$ and let $A^\top = [a_1, \ldots, a_n, \ldots, a_m]$ where we assume (WLOG) that the matrix $[a_1, \ldots, a_n]$ is non-singular (i.e., the left-hand-side of the first n constraints of $Ax \leq b$ are linearly independent). Let $B = ([a_1, \ldots a_n])^{-1}$. Show that $\{c \in \mathbb{R}^n | \exists y \geq 0, BA^\top y = c\} = \mathbb{R}^n$ if and only if P is bounded.

(c) (1pt) Suppose $\text{rank}(A) = n$ and B is as defined above. Let $\{e^1, e^2, \ldots, e^n\}$ be the n-standard unit vectors and $-\mathbf{1} = [-1, -1, \ldots, -1]^\top$. Show that if P is bounded, then all these vectors lie in the conic combinations of a subset of columns of $BA^\top$ where the cardinality of this subset is at most $2n$.

(d) (3pt) Show that if P is bounded and $m > 2n$, then one can always select a constraint in the system $Ax \leq b$ such that removing this inequality leaves the resulting polyhedron bounded.