# ACO Comprehensive Exam Spring 2023

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## 1 Algorithms

**Problem (Super Mario Secret Level)**

Super Mario unlocked a new world, where he runs with Bowser on a weighted, directed graph. Each directed edge has two positive weights $(m_e, b_e)$ corresponding to Mario and Bowser, respectively. When the arch-enemies traverse an edge, they multiply their respective number of coins by the weight for them on that edge. For example, if they are at vertex $u$ and they each have ten coins, when traversing edge $e = u \to v$ with weights $(2, 1.5)$ Mario ends at vertex $v$ with 20 coins and Bowser gets there with 15 coins. They always run together.
You are given the directed graph $\{G = (V, E), (m_e, b_e),\ e \in E\}$. Both Mario and Bowser are at vertex $s \in V$, and have one coin each.

(a) (1 points) Let $\ell_e = \log(b_e) - \log(m_e)$. Show that if Mario and Bowser take the path $\mathcal{P} = \{e_1, e_2, e_3, \ldots, e_k\}$ ending at vertex $v \in V$, then

$$\frac{\text{Number of coins Bowser has at vertex } v}{\text{Number of coins Mario has at vertex } v} = 2^{\sum_{i=1}^{k} \ell_i}.$$

(b) (3 points) You are told that $b_e > 1$ for all $e \in E$ (i.e., Bowser will always increase his number of coins!). Mario will be able to rescue the princess only if Bowser gets to the special vertex $t \in V$ with less than 100 coins. Design an algorithm that takes as input the weighted graph $G$ and outputs whether or not Mario can rescue the princess.

(c) (6 points) In the bonus level, you are given a graph with positive edge weights, not necessarily bigger than one. This time, Mario will rescue the princess iff there is an *infinite* path such that the ratio of the number of coins of Mario gets to the number of coins Bowser gets tends to infinity. Design an algorithm to decide if it is possible for Mario to rescue the princess.

To earn full credit, you must describe your algorithm in detail and justify its correctness. You should also state and analyze its runtime, which should be polynomial in the size of the graph.

**Solution:**
**(a)** After traversing $\mathcal{P}$, Mario will have $\Pi_{i=1}^{k} m_i$ many coins, and Bowser will have $\Pi_{i=1}^{k} b_i$. Thus,

$$\frac{\text{Number of coins Bowser has at vertex } v}{\text{Number of coins Mario has at vertex } v} = \frac{\Pi_{i=1}^{k} m_i}{\Pi_{i=1}^{k} b_i} = 2^{\sum_{i=1}^{k} \ell_i}.$$

**(b)** We want a path $\mathcal{P}$ from $s$ to $t$ such that $\Pi_{e \in \mathcal{P}} b_e < 100$. This is equivalent to

$$\sum_{e \in \mathcal{P}} \log(b_e) < \log(100).$$

which is a standard min-path problem. Note that the weights satisfy $b_e > 1$ so $\log(b_e) > 1$ so we can use Dijkstra's algorithm. Mario will be able to rescue the princess if and only if the length of the shortest path from $s$ to $t$ is less than 100.

**(c)** Now we set edge weights equal to $\ell_e = \log(m_e) - \log(b_e)$. Using the result from part (a), we need to find an infinite path $\mathcal{P}$ such that

$$\sum_{\mathcal{P}} \ell_i \to -\infty$$

which is equivalent to finding a cycle of negative weight! We can, e.g., run Bellman-Ford on $G$ to check for such a cycle.

## 2 Graph Theory

*Question.* For positive integers $s, t$, let $R(s,t)$ denote the least integer $n$ such that any graph on $n$ vertices contains an independent set of size $s$ or a clique of order $t$. For a family of graphs $\mathcal{F}$ we say that a graph $G$ is $\mathcal{F}$-free if no induced subgraph of $G$ is isomorphic to a member of $\mathcal{F}$. For parts (1) and (2), let $s,t$ be integers with $s \geq 3$ and $t \geq 3$.

(1) Let $G$ be a $\{K_{1,s}, K_{t+1}\}$-free graph. Show that $\chi(G) \leq R(s,t) - 1$.
(Hint: first show that $\Delta(G) \leq R(s,t) - 1$.)

(2) Show that there exists $\{K_{1,s}, K_{t+1}\}$-free graph $G$ such that $\chi(G) \geq \frac{R(s,t+1)-1}{s-1}$.

(3) Suppose $G$ is $\{K_{1,3}, K_4, W_5\}$-free, where $W_5$ denote the wheel on 6 vertices. Show $\chi(G) \leq 4$.
($W_5$ is the graph containing a 5-cycle and one additional vertex connected to all vertices of the 5-cycle).

*Solution.* (1) First, we show that $\Delta(G) \leq R(s,t) - 1$. Let $x \in V(G)$. If $d(x) \geq R(s,t)$, then by definition of $R(s,t)$, the subgraph of $G$ induced by $N(x)$ contains an independent set $S$ of size $s$ or a clique $T$ of order $t$. Now $G[S \cup \{x\}]$ is isomorphic to $K_{1,s}$ or $G[V(T) \cup \{x\}]$ is a clique of order $t+1$, a contradiction.
Thus, by Brooks' theorem, $\chi(G) \leq R(s,t)$, with equality iff $G \cong K_{R(s,t)}$ or $G$ is an odd cycle. If $G \cong K_{R(s,t)}$ then $G$ contains $K_{t+1}$ (as $R(s,t) > t+1$), a contradiction. If $G$ is an odd cycle then $\chi(G) = 3 < R(s,t)$. Hence, $\chi(G) \leq R(s,t) - 1$.

(2) By definition of $R(s,t+1)$, there exists a graph $G$ of order $R(s,t+1) - 1$ with no independent set of size $s$ (i.e. $\alpha(G) \leq s-1$) and no clique of order $t+1$. Thus $G$ is $\{K_{1,s}, K_{t+1}\}$-free and

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \geq \frac{R(s,t+1)-1}{s-1}.$$

(3) Note that $R(3,3) = 6$; so by the proof of (1), $\Delta(G) \leq 5$. Further note that $C_5$ is the only graph on 5 vertices with no independent set of size 3 and no clique of order 3. Thus, if $x$ is a vertex of degree 5 in $G$, then $G[N[x]] \cong W_5$, a contradiction. Hence, $\Delta(G) \leq 4$. By Brooks' theorem, $\chi(G) \leq 4$ unless $G \cong K_5$ or $G$ is an odd cycle. But $G \not\cong K_5$ as $G$ is $K_4$-free, and if $G$ is an odd cycle then $\chi(G) = 3 < 4$.

# 3 Linear Inequalities

Suppose we have a collection of closed, finite intervals of the real line $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$, with $I_i \subset \mathbb{R}$.

(i) (3 points) We want to find the maximum cardinality sub-collection of intervals that are disjoint, i.e., each point on the real line lines in at most one interval in the collection in which there is at most one of the chosen intervals. Write a linear program for this so that the extreme points of the feasible region are exactly subsets of disjoint intervals.

(ii) (7 points) Suppose that every point on the real line lies in at most $k$ intervals in $\mathcal{I}$. Show that $\mathcal{I}$ can be partitioned into $I_1, I_2, \ldots, I_k$ such that each $I_k$ is a collection of non-overlapping intervals.

**Solution.**

(i) Write the constraint matrix where each column represents an interval, and each row represents a member of the partition of $\mathbb{R}$ created by intersecting all the intervals. Every column will have a consecutive set of ones. The constraint for selecting a set of non-intersecting columns can be written as: $Mx \leq 1, x \in \{0, 1\}$. But note that $M$ is a TU matrix, using the Ghouila-Houri condition to partition the rows of this constraint matrix (and, therefore this can be solved in polynomial time). The extreme points of the linear program are therefore integral as well, each representing a subset of non-overlapping intervals.

(ii) We can derive this using the integer decomposition property. Consider $P = \{x | x \geq 0, Mx \leq 1\}$. We note that $x' = \frac{1}{k}$ is feasible for this polytope, since each point $p$ has at most $k$ intervals overlapping it. But $M$ is TU, and $P$ has the integer decomposition property. Therefore, $y = kx' = \mathbf{1}$, we have $My \leq k\mathbf{1}$. Therefore, $y$ can be written as $x_1 + \ldots + x_k$ where $x_i$ is integral $\forall 1 \leq i \leq k$ and $x_i \in P$. This gives the desired decomposition.