1. Computability, Complexity and Algorithms

Problem (Olympic training).

Two GeorgiaTech runners are training to compete in the next Olympic Games marathon competition. They need a route to train on. They got a map of Atlanta divided into N neighborhoods connected by $N-1$ roads in a way such that it is possible to go from any neighborhood to another. They noticed the map also shows M trails connecting some neighborhoods and decide to set a route to train using roads and trails.

(a) The runners find out that trails are nicer to exercise and decide to choose a running map that connects all neighborhoods and has the longest total distance over trails, without creating cycles. Design an algorithm that finds such optimal route, or report no if it does not exist.

Your input is a list L_1 of neighborhoods, a list L_2 of pairs of neighborhoods connected by a road, and a list L_3 of pairs of neighborhoods connected by a trail. Furthermore, for each trail a nonnegative number is given, representing its length. Make sure to justify the correctness of your design and state and analyze its running time.

(b) Now that the team knows the city, they agree to set a new route that does not visit a neighborhood nor travel a road or trail twice. The training route can start at any neighborhood, does not need to visit every neighborhood, but must end where it started. Also, for training purposes, they need to pick a route with an even total number of roads plus trails. The issue is that a rival team hears of the plan and set to block certain trails (roads cannot be blocked!) in a way that it is impossible to build a valid training route. There is a cost c_i to block trail i, for each $1 \leq i \leq M$. Design an algorithm that finds the smallest total cost needed to block some trails such that no training route exists satisfying the above requirements.

As input you get the same lists $L_i, i = 1, 2, 3$, from part **(a)**, and the cost $c_i > 0$ for blocking the i^{th} trail. *You also notice that every neighborhood has at most four roads/trails connecting to it.* Make sure to state and briefly analyze the running time of your design in terms of N and M.

2. Theory of Linear Inequalities

Problem. Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ be a polytope and let V be the set of vertices of P. Let S be a subset of V with the property that, for every $v \in V \setminus S$, the set S contains all the vertices adjacent to v. Recall, two vertices are adjacent if the segment joining them is an edge (face of dimension 1) of P. Consider the following algorithm.

1. Compute a vertex v of P minimizing $\{c^T x : x \in P\}$. If $v \in S$, return v.
2. Otherwise return a vertex v' of P minimizing $\{c^T x : x \text{ is adjacent to } v\}$.

Show the following for the algorithm.

1. (3 points) The algorithm can be implemented in polynomial time. (Here A, b is the input to the problem. $|V|$ and $|S|$ can be exponential in size of input. Moreover you are also given a polynomial time algorithm to check if $x \in conv(S)$ for any x.)

2. (5 points) Show that the algorithm solves the problem $\min\{c^T x : x \in S\}$.

3. (2 points) Give a polynomial algorithm to optimize a linear function over the set of 0,1 vectors in $\mathbb{R}^n$ with an even number of 1's.

3. Graph Theory

Problem. Let G be a connected graph and assume that G admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.

(1) Use a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow in G to construct a simple cycle cover of G, i.e., a collection $\mathcal{C}$ of cycles in G such that every edge of G appears in at least one but at most two cycles in $\mathcal{C}$.

(2) Show that the $\mathcal{C}$ in (1) may be chosen to satisfy $\sum_{C \in \mathcal{C}} |E(C)| \leq |E(G)| + |V(G)| - 1$.