SUBDIVISIONS OF COMPLETE GRAPHS

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SUBDIVISIONS OF COMPLETE GRAPHS

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To my parents.

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SUMMARY

A subdivision of a graph G, also known as a topological G and denoted by TG, is a graph obtained from G by replacing certain edges of G with internally vertex-disjoint paths. This dissertation has two parts. The first part studies a structural problem and the second part studies an extremal problem.

In the first part of this dissertation, we focus on TK_5, or subdivisions of K_5. A well known theorem of Kuratowski in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of K_5 or $K_{3,3}$. Wagner proved in 1937 that if a graph other than K_5 does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans and, independently, Seymour conjectured in the 1970s that if a graph does not contain any subdivision of K_5 then it is planar or it admits a cut of size at most 4. In this dissertation, we give a proof of the Kelmans-Seymour conjecture. We also discuss several related results and problems.

The second part of this dissertation concerns subdivisions of large cliques in C_4-free graphs. Mader conjectured that every C_4-free graph with average degree d contains TK_l with $l = \Omega(d)$. Komlós and Szemerédi reduced the problem to expanders and proved Mader's conjecture for n-vertex expanders with average degree $d < \exp(\log^{1/8} n)$. In this dissertation, we show that Mader's conjecture is true for n-vertex expanders with average degree $d < n^{3/10}$, which improves Komlós and Szemerédi's quasi-polynomial bound to a polynomial bound. As a consequence, we show that every C_4-free graph with average degree d contains a TK_l with $l = \Omega(d/(\log d)^c)$ for any $c > 3/2$. We note that Mader's conjecture has been recently verified by Liu and Montgomery.

CHAPTER 1

INTRODUCTION

We begin with some basic notation and terminology for graphs. A (simple) graph G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a set and $E(G)$ is a set of 2-element subsets of $V(G)$. A *vertex* is an element of $V(G)$ and an *edge* is an element of $E(G)$. A graph is *finite* if it contains finite number of vertices. In this dissertation, we only focus on finite graphs.

Given a graph G, an edge $\{u, v\}$ of G can also be written as uv. Two vertices u, v of G are *adjacent* in G if $uv \in E(G)$. A vertex u is a *neighbor* of a vertex v in G if u is adjacent to v. For any $u \in V(G)$, the *neighborhood* of u is the set of neighbors of u in G, denoted as $N_G(u)$. The *degree* of a vertex u is the size of its neighborhood, denoted as $deg_G(u)$. When understood, the reference to G may be dropped. The *maximum degree* $\Delta(G)$ of a graph G is the maximum of degree of a vertex in G. The *minimum degree* $\delta(G)$ of a graph G is the minimum of degree of a vertex in G. The *average degree* $d(G)$ of a graph G is the average of degree of a vertex in G. A vertex v of G is *incident* to an edge e of G if $v \in e$. A complete graph on n vertices, denoted as K_n is the graph of n vertices such that every pair of vertices are adjacent. A graph $G = (V, E)$ is called *r-partite* if V admits a partition into r classes such that every edge is adjacent to two vertices in different classes: vertices in the same partition class must not be adjacent. Instead of "2-partite" one usually says *bipartite*. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete*. Moreover, K_4^- is the graph obtained from K_4 with a single edge removed and $K_{3,3}$ is the complete bipartite graph with two partitions of size 3.

Given two graphs S and G, we say S is a *subgraph* of G if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$, denoted as $S \subseteq G$. We may view $S \subseteq V(G)$ as a subgraph of G with vertex set S and no edges. For $S \subseteq G$, the subgraph of G induced by $V(S)$, denoted as $G[S]$, is the graph with $V(G[S]) = V(S)$ and $E(G[S]) = \{uv \in E(G) : u, v \in V(S)\}$. For $S \subseteq G$ let

$N_G(S) = \{x \in V(G) \setminus V(S) : N_G(x) \cap V(S) \neq \emptyset\}$. When understood, the reference to G may be dropped.

For $S \subseteq E(G)$, $G - S$ denotes the graph obtained from G by deleting all edges in S; and for $K, L \subseteq G$, $K - L$ denotes the graph obtained from K by deleting $V(K \cap L)$ and all edges of K incident with $V(K \cap L)$.

A *separation* in a graph G consists of a pair of subgraphs G_1, G_2 of G, denoted as (G_1, G_2), such that $E(G_1) \cup E(G_2) = E(G)$, $E(G_1 \cap G_2) = \emptyset$, and $E(G_1) \cup (V(G_1) \setminus V(G_2) \neq \emptyset \neq E(G_2) \cup (V(G_2) \setminus V(G_1))$. The *order* of this separation is $|V(G_1) \cap V(G_2)|$, and (G_1, G_2) is said to be a k*-separation* if its order is k. Thus, a set $S \subseteq V(G)$ is a k*-cut* (or a *cut* of size k) in G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = S$ and $V(G_1 - S) \neq \emptyset \neq V(G_2 - S)$. If $v \in V(G)$ and $\{v\}$ is a cut of G, then v is said to be a *cut vertex* of G. For a positive interger k, we say that G is k*-connected* if G has no cut of size less than k. For $A \subseteq V(G)$ and for a positive integer k, we say that G is (k, A)*-connected* if, for any cut S with $|S| < k$, every component of $G - S$ contains a vertex from A.

A *path* is a non-empty graph $P = (V(P), E(P))$ where $V(P)$ consists of distinct vertices $v_0, v_1, ..., v_n$ and $E(P) = \{v_0v_1, v_1v_2, ..., v_{n-1}v_n\}$. The *length* of a path is the number of edges it contains. Given a path P in a graph and $x, y \in V(P)$, xPy denotes the subpath of P between x and y (inclusive). The *ends* of the path P are the vertices of the minimum degree in P, and all other vertices of P (if any) are its *internal* vertices. A path P with ends u and v (or an u-v path) is also said to be *from u to v* or *between u and v*. A collection of paths are said to be *independent* if no vertex of any path in this collection is an internal vertex of any other path in the collection. The *distance* between two vertices u and v in a graph G is the minimum length of a u-v path in G.

A *cycle* is a non-empty graph $C = (V(C), E(C))$ where $V(C)$ consists of distinct vertices $v_0, v_1, ..., v_n$ and $E(C) = \{v_0v_1, v_1v_2, ..., v_{n-1}v_n, v_nv_0\}$. The *length* of a cycle is the number of edges it contains. The *girth* of a graph G, denoted as $g(G)$, is the minimum length of a cycle contained in G.

Let G be a graph. Let $K \subseteq G$, $S \subseteq V(G)$, and T a collection of 2-element subsets of $V(K) \cup S$. Then $K + (S \cup T)$ denotes the graph with vertex set $V(K) \cup S$ and edge set $E(K) \cup T$, and if $T = \{\{x, y\}\}$ we write $K + xy$ instead of $K + \{\{x, y\}\}$.

For any positive integer k, let $[k] := \{1, \dots, k\}$. A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \dots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$ when $k = 0$) such that

(a) for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,

(b) for $i \in [k]$, $|N(A_i)| \leq 3$, and

(c) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ may be drawn in a closed disc D with no pair of edges crossing such that, for each A_i with $|N(A_i)| = 3$, $N(A_i)$ induces a facial triangle in $p(G, \mathcal{A})$.

If, in addition, $b_1, \dots, b_n$ are vertices of G such that $b_i \notin A_j$ for any $i \in [n]$ and $j \in [k]$ and $b_1, \dots, b_n$ occur on the boundary of the disc D in that cyclic order, then we say that $(G, \mathcal{A}, b_1, \dots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_1, \dots, b_n)$ is 3-planar. If there is no need to specify the order of $b_1, \dots, b_n$ then we simply say that $(G, \{b_1, \dots, b_n\})$ is 3-planar. When $\mathcal{A} = \emptyset$, we say that $(G, b_1, \dots, b_n)$ and $(G, \{b_1, \dots, b_n\})$ are *planar*. An *apex* graph is a graph that can be made planar by the removal of a single vertex.

Given a graph F, an *F-subdivision* or a *subdivision* of F is a graph H obtained from F by replacing edges of F with paths through new vertices of degree 2, denoted as TF. If G contains an F-subdivision as a subgraph, we say F is a *topological minor* of G and G contains TF. Furthermore, the vertices in TF that correspond to the vertices of F are said to be its *branch* vertices. In particular, TK_5 denotes a subdivision of K_5, and the vertices in a TK_5 of degree four are its branch vertices.

A (proper) k-*coloring* of a graph $G = (V, E)$ is a map $c : V \to [k]$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The chromatic number $\chi(G)$ of G is the smallest integer k such that G has a k-coloring.

For additional notations and background on graph theory, the readers are referred to Diestel's text [1].

This dissertation studies the structural and extremal aspects of subdivisions of complete graphs. In the next chapter, we focus on TK_5, or subdivisions of K_5. A well known theorem of Kuratowski in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of K_5 or $K_{3,3}$. Wagner proved in 1937 that if a graph other than K_5 does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans and, independently, Seymour conjectured in the 1970s that if a graph does not contain any subdivision of K_5 then it is planar or it admits a cut of size at most 4. In the next chapter, we give a proof of the Kelmans-Seymour conjecture by proving the following

Theorem 1.0.1 *Every* 5*-connected non-planar graph contains* TK_5*.*

We also discuss several related results and problems.

In Chapter 3, we study subdivisions of large cliques in C_4-free graphs. Mader conjectured that every C_4-free graph with average degree d contains TK_l with $l = \Omega(d)$. Komlós and Szemerédi reduced the problem to expanders and proved Mader's conjecture for n-vertex expanders with average degree $d < \exp(\log^{1/8} n)$. In Chapter 3, we show that Mader's conjecture is true for n-vertex expanders with average degree $d < n^{3/10}$ by showing the following

Theorem 1.0.2 *Let* $0 < \epsilon_1 < 1$ *and* $\epsilon_2 > 0$*. Let* G *be a* C_4*-free bipartite* $(\epsilon_1, \epsilon_2 d^2)$*-expander on* n *vertices with average degree* d *and* $\delta(G) \geq d/2$*. Suppose* $n \geq d^c$ *for some constant* $c > 10/3$*. Then* G *contains* TK_l *with* $l = \Omega(d)$*.*

This improves Komlós and Szemerédi's quasi-polynomial bound to a polynomial bound. As a consequence, we show that every C_4-free graph with average degree d contains a TK_l

with $l = \Omega(d/(\log d)^c)$ for any $c > 3/2$. We note that Mader's conjecture has been recently verified by Liu and Montgomery.

CHAPTER 2

K_5-SUBDIVISIONS IN 5-CONNECTED NONPLANAR GRAPHS

In this chapter, we study K_5-subdivisions in 5-connected nonplanar graphs. A well known theorem of Kuratowski in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of K_5 or $K_{3,3}$. Wagner proved in 1937 that if a graph other than K_5 does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans and, independently, Seymour conjectured in the 1970s that if a graph does not contain any subdivision of K_5 then it is planar or it admits a cut of size at most 4. In this chapter, we give a proof of the Kelmans-Seymour conjecture. We also discuss several related results and problems.

2.1 Introduction

In 1930, Kuratowski [2] prove the following well known result.

Theorem 2.1.1 *A graph is planar if, and only if, it does not contain TK_5 or $TK_{3,3}$.*

A simple application of Euler's formula for planar graphs shows that, for $n \geq 3$, if an n-vertex graph has at least $3n-5$ edges then it must be nonplanar and, hence, contains TK_5 or $TK_{3,3}$. Dirac [3] conjectured that for $n \geq 3$, if an n-vertex graph has at least $3n-5$ edges then it must contain TK_5. This conjecture was also reported by Erdős and Hajnal [4]. Kelmans [5] showed that a minimal counterexample to Dirac's conjecture must be 5-connected. Kézdy and McGuiness [6] showed that a minimal counterexample to Dirac's conjecture must be 5-connected and contains K_4^- (obtained from the complete graph K_4 by deleting an edge). After some partial results in [7, 8, 9, 10], Dirac's conjecture was proved by Mader [11], where he also showed that every 5-connected n-vertex graph with at least $3n-6$ edges contains TK_5 or K_4^-.

Seymour [12] (also see [11, 10]) and, independently, Kelmans [5] made the following.

Conjecture 2.1.2 *Every 5-connected nonplanar graph contains TK_5.*

Thus, the Kelmans-Seymour conjecture implies Mader's theorem. This conjecture is also related to several interesting problems, which we will discuss later.

He, Wang and Yu [13, 14, 15] produced lemmas needed for proving this Kelmans-Seymour conjecture, and we are now ready to prove Theorem 1.0.1 in this dissertation.

The starting point of our work is the following result of Ma and Yu [16, 17]: Every 5-connected nonplanar graph containing K_4^- has a TK_5. This result, combined with the result of Kézdy and McGuiness [6] on minimal counterexamples to Dirac's conjecture, gives an alternative proof of Mader's theorem. Also using this result, Aigner-Horev [18] proved that every 5-connected nonplanar apex graph contains TK_5. A simpler proof of Aigner-Horev's result using discharging argument was obtained by Ma, Thomas and Yu, and, independently, by Kawarabayashi, see [19].

The reminder of this chapter is organized as follows. In the next section, we discuss several related problems. We give a brief sketch of the proof of Theorem 1.0.1 in Section 2.3. We will need a number of results from [13, 14, 15], which are given in Section 2.4. In Section 2.5, we derive a simplified version of a result on disjoint paths from [20, 21, 22], which will be used several times in Section 2.6. For each subgraph T of H with $v \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, we will associate to it a quadruple (T, S_T, A, B), where, roughly, $A \cap B = \emptyset$, $H - S_T = A \cup B$, and H has no edge between A and B. (A precise definition of a quadruple is given in Section 2.6.) In Section 2.6, we prove some basic properties of quadruples, and take care of two special cases involving quadruples (using disjoint paths results from Section 2.5). In Section 2.7, we take care of other cases involving quadruples. We complete the proof of Theorem 1.0.1 in Section 2.8.

2.2 Related Problems

Theorem 1.0.1 implies that if a graph contains no TK_5 then it is planar, or admits a cut of size at most 4. This is a step towards a more useful structural description of the class of graphs containing no TK_5. There is a nice result for graphs containing no $TK_{3,3}$ due to Wagner [29]: Every such graph is planar, or is a K_5, or admits a cut of size at most 2.

Mader [11] conjectured that every simple graph with minimum degree at least 5 and no K_4^- contains TK_5, and he also asked the following.

Question 2.2.1 *Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain K_4^-, $K_{2,3}$, or TK_5?*

In a recent paper [19], it is shown that an affirmative answer to Question 2.2.1 implies the Kelmans-Seymour conjecture. As an independent approach to resolve the Kelmans-Seymour conjecture, Kawarabayashi, Ma and Yu planned to find a contractible cycle in a 5-connected nonplanar graph containing no K_4^- or $K_{2,3}$, and then use such a cycle to find a TK_5 by applying augmenting path arguments. This plan (if successful), combined with the results in [17, 19], would give an alternative (and cleaner) solution to the Kelmans-Seymour conjecture.

One of the motivations for us to work on the Kelmans-Seymour conjecture was the following conjecture of Hajós (see e.g., [30]) which, if true, would generalize the Four Color Theorem.

Conjecture 2.2.2 *Graphs containing no TK_5 are 4-colorable.*

It is known that Conjecture 2.2.2 holds for graphs with large girth (see Kühn and Osthus [31]). Let G be a possible counterexample to Conjecture 2.2.2 with $|V(G)|$ minimum. Then our result on the Kelmans-Seymour conjecture implies that G has connectivity at most 4. By a standard coloring argument, it is easy to show that G must be 3-connected. It is shown in [32] that G must be 4-connected. It is further shown in [33] that for every 4-cut

T of G, $G-T$ has exactly two components. The work in [32, 33] suggests that G should be "close" to being 5-connected.

Hajós actually made a more general conjecture in the 1950s: For any positive integer k, every graph containing no TK_{k+1} is k-colorable. This is easy to verify for $k \leq 3$ (see [34]), and disproved in [35] for $k \geq 6$. However, it remains open for $k = 4$ (Conjecture 2.2.2) and $k = 5$. Thomassen [30] pointed out connections between Hajós' conjecture and Ramsey numbers, maximum cuts, and perfect graphs. We refer the reader to [30] for other work and references related to Hajós' conjecture and topological minors.

In fact, Erdős and Fajtlowicz [36] showed that the above general Hajós' conjecture for $k \geq 6$ fails for almost all graphs. Let $H(n) := \max\{\chi(G)/\sigma(G) : G$ is a graph with $|V(G)| = n\}$, where $\chi(G)$ denotes the chromatic number of G and $\sigma(G)$ denotes the largest t such that G contains TK_t. Erdős and Fajtlowicz [36] showed that $H(n) = \Omega(\sqrt{n}/\log n)$, and conjectured that $H(n) = \Theta(\sqrt{n}/\log n)$. This conjecture was verified by Fox, Lee and Sudakov [37], by studying $\sigma(G)$ in terms of independence number $\alpha(G)$. The following conjecture of Fox, Lee and Sudakov [37] is interesting.

Conjecture 2.2.3 *There is a constant $c > 0$ such that every graph G with $\chi(G) = k$ satisfies $\sigma(G) \geq c\sqrt{k \log k}$.*

A key idea in [16, 17, 13, 14, 15] for finding TK_5 in graphs containing K_4^- is to find a nonseparating path in a graph that avoids two given vertices. Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ such that $\{x_1, x_2, y_1, y_2\}$ induces a K_4^- in which x_1, x_2 are of degree 3. We used an induced path X in G between x_1 and x_2 such that $G-X$ is 2-connected and $\{y_1, y_2\} \not\subseteq V(X)$, and in certain cases we need X to contain a special edge at x_1 (for example, in Section 2.8, $x_1 = x$ is the special vertex representing the contraction of M). If we could find such X that $G-X$ is 3-connected then our proofs would have been much simpler. This is related to the following conjecture of Lovász [38].

Conjecture 2.2.4 *There exists an integer valued function $f(k)$ such that for any $f(k)$-connected graph G and for any $A \subseteq V(G)$ with $|A| = 2$, there exist vertex disjoint sub-*

graphs G_1, G_2 of G such that $V(G_1) \cup V(G_2) = V(G)$, G_1 is a path between the vertices in A, and G_2 is k-connected.

A classical result of Tutte [39] implies $f(1) = 3$. That $f(2) = 5$ was proved by Kriesell [40] and, independently, by Chen, Gould and Yu [41]. Despite much effort from the research community, Conjecture 2.2.4 remains open for $k \geq 3$. Variations of Conjecture 2.2.4 for $k = 2$ are used in [16, 17, 13, 14, 15] to resolve the Kelmans-Seymour conjecture. An edge version of Conjecture 2.2.4 was conjectured by Kriesell and proved by Kawarabayashi *et al.* [42]. Thomassen [43] conjectured a statement that is more general than Conjecture 2.2.4 by allowing $|A| \geq 2$ and requiring $A \subseteq V(G_1)$ and G_1 be k-connected.

2.3 Proof sketch of Theorem 1.0.1

We now briefly describe the process for proving Theorem 1.0.1. For a more detailed version, we recommend the reader to read Section 2.8 first, which should also give motivation to some of the technical lemmas listed in Sections 2.4, 2.5, 2.6 and 2.7.

Suppose G is a 5-connected non-planar graph not containing K_4^-. We fix a vertex $v \in V(G)$, and let M be a maximal connected subgraph of G such that $v \in V(M)$, G/M (the graph obtained from G by contracting M) is nonplanar, G/M contains no K_4^-, and G/M is 5-connected (i.e., M is contractible). Note that $V(M) = \{v\}$ is possible. Let x denote the vertex of $H := G/M$ resulting from the contraction of M. Then, for each subgraph T of H with $v \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, H/T is planar, or H/T contains K_4^-, or H/T is not 5-connected. If, for some T, H/T is planar or contains K_4^- then we can find a TK_5 in G using results from [13, 14, 15]. Thus, in this dissertation, our main work is to deal with the final case: for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, H/T is nonplanar, H/T contains no K_4^-, and H/T is not 5-connected. In this case, there exists $S_T \subseteq V(H)$ such that $V(T) \subseteq S_T$, $|S_T| = 5$ or $|S_T| = 6$, and $H - S_T$ is not connected. We will be using such cuts to divide the graph into smaller parts and use them to find a special TK_5 in H. The reason to also include the case $T \cong K_3$ is to avoid the situation when $T \cong K_2$, $|S_T| = 5$, and $H - S_T$ has exactly two components, one of which is trivial. This does not cause problem when $T \cong K_3$, as the graph H would then contain K_4^-, and we could use results from [13, 14, 15].

2.4 Previous results

In this section, we list a number of previous results which we will use as lemmas in our proof of Theorem 1.0.1. We begin with the main result of [16, 17].

Lemma 2.4.1 *Every* 5*-connected nonplanar graph containing* K_4^- *has a* TK_5.

We also need the main result of [14] to take care of the case when the vertex x in $H = G/M$ (see Section 2.3) is a degree 2 vertex in a K_4^- (which is y_2 in the lemma below).

Lemma 2.4.2 *Let* G *be a* 5*-connected nonplanar graph and* $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ *such that* $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ *with* $y_1y_2 \notin E(G)$. *Then one of the following holds:*

(i) G *contains a* TK_5 *in which* y_2 *is not a branch vertex.*

(ii) $G - y_2$ *contains* K_4^-.

(iii) G *has a* 5*-separation* (G_1, G_2) *such that* $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, *and* G_2 *is the graph obtained from the edge-disjoint union of the* 8*-cycle* $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ *and the* 4*-cycle* $b_1b_2b_3b_4b_1$ *by adding* y_2 *and the edges* y_2b_i *for* $i \in [4]$.

(iv) *For* $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ *contains* TK_5.

To deal with conclusion (iii) of Lemma 2.4.2, we need Proposition 1.3 from [13] in which a plays the role of y_2 in Lemma 2.4.2.

Lemma 2.4.3 *Let* G *be a 5-connected nonplanar graph,* (G_1, G_2) *a 5-separation in* G, $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ *such that* G_2 *is the graph obtained from the edge-disjoint* *union of the* 8*-cycle* $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ *and the* 4*-cycle* $b_1b_2b_3b_4b_1$ *by adding* a *and the edges* ab_i*,* $i \in [4]$*. Suppose* $|V(G_1)| \geq 7$*. Then, for any* $u_1, u_2 \in N(a) - \{b_1, b_2, b_3\}$*,* $G - \{av : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$ *contains* TK_5*.*

Next we list a few results from [13, 14, 15]. For convenience, we state their versions from [15]. First, we need Theorem 1.1 in [15] to take care of the case when the vertex x in $H = G/M$ (see Section 2.3) is a degree 3 vertex in a K_4^- (which is x_1 in the lemma below).

Lemma 2.4.4 *Let* G *be a* 5*-connected nonplanar graph and* $x_1, x_2, y_1, y_2 \in V(G)$ *be distinct such that* $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ *and* $y_1y_2 \notin E(G)$*. Then one of the following holds:*

(i) *G contains a TK_5 in which x_1 is not a branch vertex.*

(ii) *$G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.*

(iii) *x_2, y_1, y_2 may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1v : v \notin \{x_2, y_1, y_2, z_0, z_1\}\}$ contains TK_5.*

When applying the next three lemmas, the vertex a will correspond to the vertex x in $H = G/M$ in Section 2.3. The following result is Lemma 2.7 in [15], which deals with 5-separations with an apex side.

Lemma 2.4.5 *Let* G *be a* 5*-connected nonplanar graph and let* (G_1, G_2) *be a* 5*-separation in* G*. Suppose* $|V(G_i)| \geq 7$ *for* $i \in [2]$*,* $a \in V(G_1 \cap G_2)$*, and* $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ *is planar. Then one of the following holds:*

(i) *G contains a TK_5 in which a is not a branch vertex.*

(ii) *$G - a$ contains K_4^-, or G contains a K_4^- in which a is of degree 2.*

The next result is Lemma 2.8 in [15], which will be used to take care of 5-cuts containing the vertices of a triangle.

Lemma 2.4.6 *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

(i) *G contains a TK_5 in which a is not a branch vertex.*

(ii) *$G - a$ contains K_4^-, or G contains a K_4^- in which a is of degree 2.*

(iii) *For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains TK_5.*

The following is Lemma 2.9 in [15].

Lemma 2.4.7 *Let G be a graph, $A \subseteq V(G)$, and $a \in A$ such that $|A| = 6$, $|V(G)| \geq 8$, $(G - a, A - \{a\})$ is planar, and G is $(5, A)$-connected. Then one of the following holds:*

(i) *$G - a$ contains K_4^-, or G contains a K_4^- in which the degree of a is 2.*

(ii) *G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $|V(G_2)| \geq 7$, $A \subseteq V(G_1)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar.*

We need Theorem 1.4 in [13]. This will be used to show that, for a quadruple (T, S_T, A, B) in $H = G/M$ with $x \in V(T)$ (see Section 2.3), x has a neighbor in A (which corresponds to $G_1 - G_2$ in the statement).

Lemma 2.4.8 *Let G be a 5-connected graph and $x \in V(G)$, and let (G_1, G_2) be a 6-separation in G such that $x \in V(G_1 \cap G_2)$, $G[V(G_1 \cap G_2)]$ contains a triangle xx_1x_2x, $|V(G_i)| \geq 7$ for $i \in [2]$. Moreover, assume that (G_1, G_2) is chosen so that, subject to $\{x, x_1, x_2\} \subseteq V(G_1 \cap G_2)$ and $|V(G_i)| \geq 7$ for $i \in [2]$, G_1 is minimal. Let $V(G_1 \cap G_2) = \{x, x_1, x_2, v_1, v_2, v_3\}$. Then $N(x) \cap V(G_1 - G_2) \neq \emptyset$, or one of the following holds:*

(i) *G contains a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(iii) *There exists* $x_3 \in N(x)$ *such that for any distinct* $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$*,* $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ *contains* TK_5*.*

(iv) *For some* $i \in [2]$ *and some* $j \in [3]$*,* $N(x_i) \subseteq V(G_1 - G_2) \cup \{x, x_{3-i}\}$*, and any three independent paths in* $G_1 - x$ *from* $\{x_1, x_2\}$ *to* v_1, v_2, v_3*, respectively, with two from* x_i *and one from* x_{3-i}*, must contain a path from* x_{3-i} *to* v_j*.*

We remark that conclusion (iv) in Lemma 2.4.8 will be dealt with in Section 2.6, using a result on disjoint paths from [20, 21, 22]. We also need Proposition 4.1 from [13] to deal with the case when H/T is planar (see Section 2.3) for some $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$.

Lemma 2.4.9 *Let* G *be a* 5*-connected nonplanar graph,* $x \in V(G)$*,* $T \subseteq G$ *such that* $x \in V(T)$*,* $T \cong K_2$ *or* $T \cong K_3$*,* G/T *is* 5*-connected and planar. Then* $G - T$ *contains* K_4^-*.*

We conclude this section with three additional results, first of which is a result of Seymour [23]; equivalent versions are proved in [24, 25, 26].

Lemma 2.4.10 *Let* G *be a graph and let* $s_1, s_2, t_1, t_2 \in V(G)$ *be distinct. Then either* G *contains disjoint paths from* s_1 *to* t_1 *and from* s_2 *to* t_2*, or* (G, s_1, s_2, t_1, t_2) *is 3-planar.*

The second result is due to Perfect [27].

Lemma 2.4.11 *Let* G *be a graph,* $u \in V(G)$*, and* $A \subseteq V(G - u)$*. Suppose there exist* k *independent paths from* u *to distinct* $a_1, \ldots, a_k \in A$*, respectively, and internally disjoint from* A*. Then for any* $n \geq k$*, if there exist* n *independent paths* $P_1, \ldots, P_n$ *in* G *from* u *to* n *distinct vertices in* A *and internally disjoint from* A *then* $P_1, \ldots, P_n$ *may be chosen so that* $a_i \in V(P_i)$ *for* $i \in [k]$*.*

The third result is due to Watkins and Mesner [28].

Lemma 2.4.12 *Let G be a 2-connected graph and let y_1, y_2, y_3 be three distinct vertices of G. Then G has no cycle containing $\{y_1, y_2, y_3\}$ if, and only if, one of the following holds:*

(i) *There exists a 2-cut S in G and there exist pairwise disjoint subgraphs D_{y_i} of $G - S$, $i \in [3]$, such that $y_i \in V(D_{y_i})$ and each D_{y_i} is a union of components of $G - S$.*

(ii) *There exist 2-cuts S_{y_i} in G, $i \in [3]$, and pairwise disjoint subgraphs D_{y_i} of G, such that $y_i \in V(D_{y_i})$, each D_{y_i} is a union of components of $G - S_{y_i}$, there exists $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*

(iii) *There exist pairwise disjoint 2-cuts S_{y_i} in G and pairwise disjoint subgraphs D_{y_i} of $G - S_{y_i}$, $i \in [3]$, such that $y_i \in V(D_{y_i})$, D_{y_i} is a union of components of $G - S_{y_i}$, and $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_{y_i} for $i \in [3]$.*

2.5 Obstruction to three paths

In order to deal with (iv) of Lemma 2.4.8, we need a result of Yu [20, 21, 22], which characterizes graphs G in which any three disjoint paths from $\{a, b, c\} \subseteq V(G)$ to $\{a', b', c'\} \subseteq V(G)$ must contain a path from b to b'. The objective of this section is to derive a much simpler version of that characterization by imposing extra conditions on G. This result will be used several times in the proofs of Lemmas 2.6.4 and 2.6.6. To state the result from [20, 21, 22], we need to describe *rungs* and *ladders*.

Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$. Suppose $\{a, b, c\} \neq \{a', b', c'\}$, and assume that G has no separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $\{a, b, c\} \subseteq V(G_1)$, and $\{a', b', c'\} \subseteq V(G_2)$. We say that $(G, (a, b, c), (a', b', c'))$ is a *rung* if one of the following holds:

(1) $b = b'$ or $\{a, c\} = \{a', c'\}$.

(2) $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar, or $c = c'$ and $(G - c, a, a', b', b)$ is 3-planar.

(3) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$ and (G,a',b',c',c,b,a) is 3-planar.

(4) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, G has a 1-separation (G_1,G_2) such that (i) $\{a,a',b,b'\} \subseteq V(G_1)$, $\{c,c'\} \subseteq V(G_2)$, and (G_1,a,a',b',b) is 3-planar, or (ii) $\{c,c',b,b'\} \subseteq V(G_1)$, $\{a,a'\} \subseteq V(G_2)$, and (G_1,c,c',b',b) is 3-planar.

(5) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and G has a separation (G_1,G_2) such that $V(G_1 \cap G_2) = \{z,b\}$ (or $V(G_1 \cap G_2) = \{z,b'\}$), and (i) (G,a,a',b',b) is 3-planar, $\{a,a',b,b'\} \subseteq V(G_1)$, $\{c,c'\} \subseteq V(G_2)$, and (G_2,c,c',z,b) (or (G_2,c,c',b',z)) is 3-planar, or (ii) (G,c,c',b',b) is 3-planar, $\{c,c',b,b'\} \subseteq V(G_1)$, $\{a,a'\} \subseteq V(G_2)$, and (G_2,a,a',z,b) (or (G_2,a,a',b',z)) is 3-planar.

(6) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u,w\}$, $V(G_c \cap M) = \{p,q\}$, $V(G_a \cap G_c) = \emptyset$, and (i) $\{a,a',b'\} \subseteq V(G_a)$, $\{c,c',b\} \subseteq V(G_c)$, and (G_a,a,a',b',w,u) and (G_c,c',c,b,p,q) are 3-planar, or (ii) $\{a,a',b\} \subseteq V(G_a)$, $\{c,c',b'\} \subseteq V(G_c)$, (G_a,b,a,a',w,u) and (G_c,b',c',c,p,q) are 3-planar.

(7) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b,b',w\}$, $V(G_c \cap M) = \{b,b',p\}$, $V(G_a \cap G_c) = \{b,b'\}$, $\{a,a'\} \subseteq V(G_a)$, $\{c,c'\} \subseteq V(G_c)$, and (G_a,a,a',b',w,b) and (G_c,c',c,b,p,b') are 3-planar.

Let L be a graph and let $R_1,\dots,R_m$ be edge disjoint subgraphs of L such that

(i) $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$ is a rung for each $i \in [m]$,

(ii) $V(R_i \cap R_j) = \{x_i,v_i,y_i\} \cap \{x_{j-1},v_{j-1},y_{j-1}\}$ for $i,j \in [m]$ with $i < j$,

(iii) for any distinct $i,j \in [m]$, if $x_i = x_j$ then $x_k = x_i$ for all $i \le k \le j$, if $v_i = v_j$ then $v_k = v_i$ for all $i \le k \le j$, and if $y_i = y_j$ then $y_k = y_i$ for all $i \le k \le j$, and

(iv) $L = (\bigcup_{i=1}^m R_i) + S$, where S consists of those edges of L each of which has both ends in $\{x_i,v_i,y_i\}$ for some $i \in [m]$.

Then $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ is a *ladder with rungs* $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$, $i \in [m]$, or simply, a *ladder along* $v_0 \dots v_m$.

By the definition of a rung, we see that a ladder $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ has three disjoint paths from $\{x_0,v_0,y_0\}$ to $\{x_m,v_m,y_m\}$.

For a sequence W, the *reduced sequence* of W is the sequence obtained from W by removing all but one consecutive identical elements. For example, the reduced sequence of $aaabcca$ is $abca$. We can now state the main result in [22].

Lemma 2.5.1 *Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$ such that $\{a,b,c\} \neq \{a',b,'c'\}$. Assume that, for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G-T$ contains some element of $\{a,b,c\} \cup \{a',b',c'\}$. Then any three disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$ must include one from b to b' if, and only if, one of the following statements holds:*

- (i) *G has a separation (G_1,G_2) of order at most 2 such that $\{a,b,c\} \subseteq V(G_1)$ and $\{a',b',c'\} \subseteq V(G_2)$.*
- (ii) *$(G,(a,b,c),(a',b',c'))$ is a ladder.*
- (iii) *G has a separation (J,L) such that $V(J\cap L) = \{w_0,\dots,w_n\}$, $(J,w_0,\dots,w_n)$ is 3-planar, $\{a,b,c\}\cup\{a',b',c'\} \subseteq V(L)$, $(L,(a,b,c),(a',b',c'))$ is a ladder along a sequence $v_0\dots v_m$, where $v_0=b$, $v_m=b'$, and $w_0\dots w_n$ is the reduced sequence of $v_0\dots v_m$.*

We may view (ii) as a special case of (iii) by letting J be a subgraph of L. In the applications of Lemma 2.5.1 in this paper, we will consider rungs and ladders in a 5-connected graph without TK_5. With such extra conditions, the rungs have much simpler structure, as given in the next two lemmas.

Lemma 2.5.2 *Let G be a 5-connected graph and (R,R') a separation in G such that $|V(R')| \geq 8$, $V(R\cap R') = \{a,b\}\cup\{a',b',c'\}$, $a\neq b$, and a',b',c' are pairwise distinct. Let R^* be obtained from R by adding the new vertex c and joining c to each neighbor of a in R with an edge, and assume $(R^*,(a,b,c),(a',b',c'))$ is a rung. Then $b=b'$, $V(R)=\{a,b,a',c'\}$ and $E(R)=\{aa',ac'\}$.*

Proof. Since a and c have the same set of neighbors in R^* and $(R^*, (a, b, c), (a', b', c'))$ is a rung, it follows from the definition of a rung that $(R^*, (a, b, c), (a', b', c'))$ is of type (1) or (2). Then, since G is 5-connected, $V(R) = \{a, b\} \cup \{a', b', c'\}$.

Suppose $(R^*, (a, b, c), (a', b', c'))$ is of type (2). By symmetry, we may assume that $c = c'$ and $(G - c, a, a', b', b)$ is 3-planar. Then $ab' \notin E(G)$ or $a'b \notin E(G)$. Hence, $\{a', b, c\}$ or $\{a, b', c\}$ would be a cut in R^* separating $\{a, b, c\}$ from $\{a', b', c'\}$, a contradiction.

So $(R^*, (a, b, c), (a', b', c'))$ is of type (1). Then, since R^* has no separation of order at most 3 separating $\{a, b, c\}$ from $\{a', b', c'\}$, we deduce that $a \neq a'$, $c \neq c'$, and $E(R) = \{aa', ac'\}$. ∎

Note that the conclusion of Lemma 2.5.2 is a special case of (i) of the next lemma.

Lemma 2.5.3 *Let G be a 5-connected graph and (R, R') a separation in G such that $|V(R')| \geq 8$, $V(R \cap R') = \{a, b, c\} \cup \{a', b', c'\}$, $\{a, b, c\} \neq \{a', b', c'\}$, and $(R, (a, b, c), (a', b', c'))$ is a rung. Then G contains TK_5 or K_4^-, or one of the following holds:*

(i) $b = b'$.

(ii) $\{a, c\} = \{a', c'\}$, $V(R) = \{a, c, b, b'\}$, *and* $E(R) = \{bb'\}$.

(iii) $V(R) - (\{a, b, c\} \cup \{a', b', c'\}) = \{v\}$ *and* $N(v) = \{a, b, c\} \cup \{a', b', c'\}$, *and either* $a = a'$ *and* $E(R - v) = \{bb', cc'\}$ *or* $c = c'$ *and* $E(R - v) = \{bb', aa'\}$.

(iv) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, $V(R) - \{a, a', b, b', c, c'\} = \{v\}$, $N(v) = \{a, a', b, b', c, c'\}$, *and* $E(R - v) = \{aa', bb', cc'\}$.

Proof. Without loss of generality, let A, B, C be disjoint paths in R from a, b, c to a', b', c', respectively. First, we consider the case when $\{a, b, c\} \cap \{a', b', c'\} \neq \emptyset$. If $b = b'$ then (i) holds; so we may assume $b \neq b'$. If $a = a'$ and $c = c'$ then, since G is 5-connected, $V(R) = \{a, b, b', c\}$; so $bb' \in E(R)$ (because of the paths A, B, C), and we have (ii). Thus by symmetry between $\{a, a'\}$ and $\{c, c'\}$, we may assume $c \neq c'$. Suppose $a = a'$. Then by the definition of a rung, $R - a$ has no disjoint paths from b, c to

c', b', respectively. So by Lemma 2.4.10, $(R - a, c, c', b', b)$ is 3-planar. Since G is 5-connected, $(R - a, c, c', b', b)$ is in fact planar. If $|V(R)| \geq 7$ then G contains TK_5 or K_4^- by Lemma 2.4.5, using the separation (R, R'). If $V(R) = \{a, b, b', c, c'\}$ then, since $(R - a, c, c', b', b)$ is planar, either $\{a, b, c'\}$ or $\{a, b', c\}$ is a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of a rung. Thus, we may assume $|V(R)| = 6$ and let $v \in V(R) - \{a, b, b', c, c'\}$. Since G is 5-connected, $N(v) = \{a, b, b', c, c'\}$. Since $(R - a, c, c', b', b)$ is planar, $bc', cb' \notin E(R)$. So $bb', cc' \in E(R)$, as otherwise $\{a, v, c\}$ or $\{a, v, b\}$ would be a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of a rung. Hence, (iii) holds.

Thus, we may assume that $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$. We need to deal with (3) – (7) in the definition of a rung. We deal with (4)–(7) in order, and treat (3) last (which is the most complicated case where we use the discharging technique).

Suppose (4) holds for $(R, (a, b, c), (a', b', c'))$. By symmetry, assume that R has a 1-separation (G_1, G_2) such that $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_1, a, a', b', b) is 3-planar. Let $V(G_1 \cap G_2) = \{v\}$. Since G is 5-connected, (G_1, a, a', b', b) is planar and $V(G_2) = \{v, c, c'\}$. Moreover, $vc, vc', cc' \in E(G)$; for otherwise R would have a separation (R_1, R_2) such that $\{a, b, c\} \subseteq V(R_1)$, $\{a', b', c'\} \subseteq V(R_2)$, and $V(R_1 \cap R_2) \in \{\{a, b, c'\}, \{a', b', c\}, \{a, b, v\}\}$. If $|V(G_1)| \geq 7$ then the assertion follows from Lemma 2.4.5, using the separation $(G_1, G_2 \cup R')$. So we may assume $|V(G_1)| \leq 6$. If $|V(G_1)| = 6$ then let $t \in V(G_1) - \{a, a', b, b', v\}$; now $N(t) = \{a, a', b, b', v\}$ and $|(N(v) - \{c, c'\}) \cap N(t)| \geq 2$ (since G is 5-connected), and hence R (and therefore G) contains K_4^-. So we may assume $V(G_1) = \{a, a', b, b', v\}$. Then $va' \in E(G)$; otherwise $N(v) = \{a, b, b', c, c'\}$ and, hence, $a'b \notin E(G)$ (as (G_1, a, a', b', b) is planar), which implies that $\{a, b', c'\}$ is a cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, a contradiction. Similarly, $va, vb, vb' \in E(G)$. Then by planarity of (G_1, a, a', b', b), we have $ab', ba' \notin E(G)$. So $aa', bb' \in E(G)$ as $\{c, v, b'\}$ and $\{a, v, c\}$ are not 3-cuts in R separating $\{a, b, c\}$ from $\{a', b', c'\}$. Thus we have (iv).

Suppose (5) holds for $(R, (a, b, c), (a', b', c'))$, and assume by symmetry that (R, a, a', b', b) is 3-planar, and R has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z, b\}$, $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_2, c, c', z, b) is 3-planar. Since G is 5-connected, $V(G_2) = \{b, z, c, c'\}$. Then $cz, cc' \in E(G)$ as otherwise, $\{a, b, c'\}$ or $\{a, b, z\}$ would be a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$. Hence, since (G_2, b, z, c', c) is planar, $bc' \notin E(G)$. Since (R, a, a', b', b) is 3-planar, (G_1, a, a', b', b) is 3-planar. Thus, the separation $(G_1, G_2 - b)$ shows that $(R, (a, b, c), (a', b', c'))$ is of type (4); so we may assume that (iv) holds by the argument in the previous paragraph.

Now suppose (6) holds for $(R, (a, b, c), (a', b', c'))$, and, by symmetry, assume that there are pairwise edge disjoint subgraphs G_a, G_c, M of R such that $R = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u, w\}$, $V(G_c \cap M) = \{p, q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, and (G_a, a, a', b', w, u) and (G_c, c', c, b, p, q) are 3-planar. Since G is 5-connected, we have $V(M) = \{p, q, u, w\}$, and (G_a, a, a', b', w, u) and (G_c, c', c, b, p, q) are planar. We may assume that $|V(G_c) - \{b, c, c', p, q\}| \leq 1$ and $|V(G_a) - \{a, a', b', u, w\}| \leq 1$, as otherwise the assertion follows from Lemma 2.4.5 with the separation $(G_c, G_a \cup M \cup R')$ or $(G_a, G_c \cup M \cup R')$. If there exists $v \in V(G_c) - \{b, c, c', p, q\}$ then, since G is 5-connected, $N(v) = \{b, c, c', p, q\}$ and $|(N(p) - \{u, w\}) \cap \{b, c, c', q\}| \geq 2$; so R (and hence G) contains K_4^-. Thus we may assume $V(G_c) = \{b, c, c', p, q\}$. Since G is 5-connected, p and q each have at least five neighbors in $G_c \cup M$. Hence, since (G_c, b, c, c', q, p) is planar, $N(p) = \{u, w, b, c, q\}$ and $N(q) = \{u, w, c, c', p\}$; so $G[\{p, q, u, w\}]$ (and hence G) contains K_4^-.

Suppose (7) holds for $(R, (a, b, c), (a', b', c'))$. Then there are pairwise edge disjoint subgraphs G_a, G_c, M of R such that $R = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b', w\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a'\} \subseteq V(G_a)$, $\{c, c'\} \subseteq V(G_c)$, and (G_a, a, a', b', w, b) and (G_c, c', c, b, p, b') are 3-planar. Since G is 5-connected, we have $V(M) = \{b, b', p, w\}$, and (G_a, a, a', b', w, b) and (G_c, c', c, b, p, b') are actually planar. If $|V(G_c)| \geq 7$ then the assertion follows from Lemma 2.4.5 with the separation $(G_c, G_a \cup$

$M \cup R'$). So we may assume $|V(G_c)| \leq 6$. If there exists $q \in V(G_c) - \{b, b', c, c', p\}$ then $N(q) = \{b, b', c, c', p\}$ (as G is 5-connected); therefore, since (G_c, c', c, b, p, b') is planar, $N(p) \subseteq \{b, b', w, q\}$, a contradiction. Thus $V(G_c) = \{b, b', c, c', p\}$ and, hence, $N(p) = \{b, b', c, c', w\}$. Similarly, by considering G_a, we may assume $N(w) = \{a, a', b, b', p\}$. Thus $G[\{b, b', p, w\}]$ (and hence G) contains K_4^-.

Finally, assume that (3) holds for $(R, (a, b, c), (a', b', c'))$. So (R, a', b', c', c, b, a) is planar (as G is 5-connected), and we may assume that R is embedded in a closed disc with no edge crossings such that a, b, c, c', b', a' occur on the boundary of the disc in clockwise order. We apply the discharging method. For convenience, let $A = \{a, b, c, a', b', c'\}$, $F(R)$ denote the set of faces of R, and f_∞ denote the outer face of R (which is incident with all vertices in A). For each $f \in F(R)$, let $d_R(f)$ denote the number of incidences of the edges of R with f, and ∂f denote the set of vertices of R incident with f. For $x \in V(R) \cup F(R)$, let $\sigma(x) = d_R(x) - 4$ be the charge of x. Note that R is connected as in R there is no separation (R_1, R_2) of order at most 3 such that $\{a, b, c\} \subseteq V(R_1)$ and $\{a', b', b'\} \subseteq V(R_2)$. Hence, by Euler's formula, $\sum_{x \in V(R) \cup F(R)} \sigma(x) = -8$.

We redistribute charges according to the following rule: For each $v \in V(R) - A$, v sends $1/2$ to each $f \in F(R)$ that is incident with v and has $d_R(f) = 3$. Let $\tau(x)$ denote the new charge for all $x \in V(R) \cup F(R)$. Then

$$\sum_{x \in V(R) \cup F(R)} \tau(x) = \sum_{x \in V(R) \cup F(R)} \sigma(x) = -8.$$

Note that we may assume $K_4^- \not\subseteq G$. Thus, each $v \in V(R) - A$ is incident with at most $\lfloor d_R(v)/2 \rfloor$ faces $f \in F(R)$ with $d_R(f) = 3$; so $\tau(v) \geq 0$ (as $d_R(v) \geq 5$). Moreover, for $f \in F(R)$, $\tau(f) \geq 0$ unless $d_R(f) = 3$ and f is incident with at least two vertices in A.

Since R has no separation (R_1, R_2) of order at most 3 such that $\{a, b, c\} \subseteq V(R_1)$ and $\{a', b', c'\} \subseteq V(R_2)$, we see that $\{a, b, c\}$ and $\{a', b', c'\}$ are independent in R. Moreover, since (R, a, a', b', c', c, b) is planar, $ab', ac', ba', bc', ca', cb' \notin E(R)$, and $d_R(v) \geq 2$ for

$v \in A$. Hence, $bb' \notin E(R)$; otherwise, since G is 5-connected, $V(R) = A$ (to avoid 4-cuts $\{a, a', b, b'\}$ and $\{b, b', c, c'\}$), which in turn would force $d_R(v) \leq 1$ for some $v \in A$.

Therefore, $d_R(f_\infty) \geq 10$, and if $f \in F(R)$ with $d_R(f) = 3$ and $|\partial f \cap A| \geq 2$ then $\partial f \cap A = \{a, a'\}$ or $\partial f \cap A = \{c, c'\}$. Hence,

$$\begin{aligned}
\sum_{x \in V(R) \cup F(R)} \tau(x) &\geq \sum_{v \in V(R)} \tau(v) + \sum_{f \in F(R), |\partial f \cap A| \geq 2} \tau(f) \\
&\geq \sum_{v \in A} (d_R(v) - 4) + (d_R(f_\infty) - 4) + \sum_{d_R(f)=3, |\partial f \cap A| \geq 2} (d_R(f) - 4) \\
&\geq (-12) + (10 - 4) + (-1) \times 2 \\
&= -8.
\end{aligned}$$

Thus, all the inequalities above hold with equality. In particular, $d_R(f_\infty) = 10$, $d(x) = 2$ for $x \in A$, and there exist $u, v \in V(R) - A$ such that $uaa'u$ and $vcc'v$ are triangles and $aa'ub'vc'cvbua$ is the outer walk of R. Since G is 5-connected and (R, a, b, c, c', b', a') is planar, $V(R) = A \cup \{u, v\}$ and $uv \in E(R)$. Hence, $G[\{b, b', u, v\}] \cong K_4^-$, a contradiction. ∎

2.6 Quadruples and special structure

As mentioned in Section 2.3, we need to deal with 5-connected graphs in which every edge or triangle at a given vertex is contained in a cut of size 5 or 6. Thus, for convenience, we introduce the following concept of quadruple.

Let G be a graph. For $x \in V(G)$, let $\mathcal{Q}_x$ denote the set of all quadruples (T, S_T, A, B), such that

(1) $T \subseteq G$, $x \in V(T)$, and $T \cong K_2$ or $T \cong K_3$,

(2) S_T is a cut of G with $V(T) \subseteq S_T$, A is a nonempty union of components of $G - S_T$, and $B = G - A - S_T \neq \emptyset$,

(3) if $T \cong K_3$ then $5 \leq |S_T| \leq 6$, and

(4) if $T \cong K_2$ then $|S_T| = 5$, $|V(A)| \geq 2$, and $|V(B)| \geq 2$.

The purpose of this section is to derive useful properties of quadruples, in particular, those (T, S_T, A, B) that minimize $|V(A)|$. We begin with a few simple properties, first of which gives a bound on $|V(A)|$.

Lemma 2.6.1 *Let G be a 5-connected graph, $x \in V(G)$, and $(T, S_T, A, B) \in \mathcal{Q}_x$. Then G contains K_4^-, or $|V(A)| \geq 5 \leq |V(B)|$.*

Proof. Suppose there exists $(T, S_T, A, B) \in \mathcal{Q}_x$ such that $|V(A)| \leq 4$ or $|V(B)| \leq 4$. We choose such $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum. Then $|V(A)| \leq 4$. Let δ denote the minimum degree of A, and let $u \in V(A)$ such that u has degree δ in A.

We may assume $\delta \geq 1$. For, suppose $\delta = 0$. If $T \cong K_3$ then, since G is 5-connected, $|N(u) \cap S_T| \geq 5$; so $G[T + u]$ contains K_4^-. Hence we may assume $T \cong K_2$. Then $|V(A)| \geq 2$. In fact, by the minimality of $|V(A)|$, $|V(A)| = 2$ and A consists of two isolated vertices. Now $G[A \cup T]$ contains K_4^-.

Case 1. $\delta = 1$.

Then $|N(u) \cap S_T| \geq 4$. Let v be the unique neighbor of u in A. Since $|V(A)| \leq 4$ and G is 5-connected, $|N(v) \cap S_T| \geq 2$. We may assume $|N(u) \cap N(v) \cap S_T| \leq 1$; for, otherwise, $G[S_T \cup \{u, v\}]$ contains K_4^-.

Suppose $|N(v) \cap S_T| \geq 3$ or $N(u) \cap N(v) \cap S_T = \emptyset$. Then $|S_T| = 6$ and, hence, $T \cong K_3$. Therefore, $|N(u) \cap V(T)| \geq 2$ or $|N(v) \cap V(T)| \geq 2$; so $G[T + u]$ or $G[T + v]$ contains K_4^-.

Hence, we may assume that $|N(v) \cap S_T| \leq 2$ and $|N(u) \cap N(v) \cap S_T| = 1$. Then, since $|V(A)| \leq 4$ and G is 5-connected, $|N(v) \cap S_T| = 2$, $|N(v) \cap V(A)| = 3$, and $|V(A)| = 4$. Let $v_1, v_2 \in V(A) - \{u, v\}$, and let $w \in N(u) \cap N(v) \cap S_T$. Since G is 5-connected, $|N(v_i) \cap S_T| \geq 3$ for $i \in [2]$.

We may assume $w \notin V(T)$; for, if $w \in V(T)$ then $|V(T) \cap N(u)| \geq 2$ or $|V(T) \cap N(v)| \geq 2$, and $G[T + \{u, v\}]$ contains K_4^-. We may also assume $w \notin N(v_i)$ for $i \in [2]$, as otherwise $G[\{u, v, w, v_i\}]$ contains K_4^-.

If $v_1v_2 \notin E(G)$ then $|N(v_i) \cap S_T| \geq 4$ for $i \in [2]$; so $|N(v_i) \cap V(T)| \geq 2$ for $i \in [2]$ (since $w \notin N(v_i)$ and $w \notin V(T)$), and hence, $G[T + \{v_1, v_2\}]$ contains K_4^-. So assume $v_1v_2 \in E(G)$. Since G is 5-connected and $w \notin N(v_i)$ for $i \in [2]$, there exists $w' \in N(v_1) \cap N(v_2) \cap S_T$. Now $G[\{v, v_1, v_2, w'\}]$ contains K_4^-.

Case 2. $\delta \geq 2$.

If $|V(A)| = 3$ then $A \cong K_3$ and, since G is 5-connected, $|N(a) \cap S_T| \geq 3$ for all $a \in V(A)$; hence, since $|S_T| \leq 6$, $G[V(A) \cup S_T]$ contains K_4^-. So assume $|V(A)| = 4$. We may further assume that A is a cycle as otherwise A contains K_4^-. Moreover, we may assume that for any $st \in E(A)$, $|N(s) \cap N(t) \cap S_T| \leq 1$; for otherwise $G[\{s, t\} \cup S_T]$ contains K_4^-. Let $A = uvwru$.

Suppose $T \cong K_2$. Then for any $st \in E(A)$, $(N(s) \cup N(t)) - V(A) = S_T$ and $|N(s) \cap N(t) \cap S_T| = 1$. Let $S_T = \{x_1, x_2, x_3, x_4, x_5\}$ and, without loss of generality, let $N(u) \cap A = \{x_1, x_2, x_3\}$ and $N(v) \cap A = \{x_3, x_4, x_5\}$. Since $(N(w) \cup N(r)) - V(A) = S_T$, $wx_3 \in E(G)$ or $rx_3 \in E(G)$. Then $G[\{u, v, w, x_3\}] \cong K_4^-$ or $G[\{r, u, v, x_3\}] \cong K_4^-$.

Now assume $T \cong K_3$. Let $S_T = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $V(T) = \{x_1, x_2, x_3\}$. We may assume $|N(a) \cap V(T)| \leq 1$ for each $a \in V(A)$, for, otherwise, $G[T + a]$ contains K_4^-. Hence, let $x_4, x_5 \in N(u)$, $x_5, x_6 \in N(v)$, and $x_6, x_4 \in N(w)$. Note that $N(r) \cap \{x_4, x_6\} \neq \emptyset$. If $x_4 \in N(r)$ then $G[\{u, w, r, x_4\}] \cong K_4^-$, and if $x_6 \in N(r)$ then $G[\{v, w, r, x_6\}] \cong K_4^-$. ∎

Next, we show that if a graph G has no contractible edge or triangle at some vertex x then every edge of G at x is associated with a quadruple in $\mathcal{Q}_x$.

Lemma 2.6.2 *Let G be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected. Then for any $ax \in E(G)$, there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $\{a, x\} \subseteq V(T')$.*

Proof. Let $T_1 = ax$. By assumption, G/T_1 is not 5-connected. So there exists a 5-cut S_{T_1} in

G with $V(T_1) \subseteq S_{T_1}$. We may assume that $G - S_{T_1}$ has a trivial component; for otherwise, let C be a component of $G - S_{T_1}$ and $D = (G - S_{T_1}) - C$. Then $(T_1, S_{T_1}, C, D) \in \mathcal{Q}_x$ is the desired quadruple.

So let $y \in V(G)$ such that y is a component of $G - S_{T_1}$. Let $T_2 := G[T_1 + y] \cong K_3$. By assumption, G/T_2 is not 5-connected. So there exists a cut S_{T_2} in G such that $V(T_2) \subseteq S_{T_2}$ and $|S_{T_2}| \in \{5, 6\}$. Let C be a component of $G - S_{T_2}$ and $D = (G - S_{T_2}) - C$. Then $(T_2, S_{T_2}, C, D) \in \mathcal{Q}_x$ is the desired quadruple. ∎

We now show that if (T, S_T, A, B) is chosen to minimize $|V(A)|$ then we may assume $T \cong K_3$.

Lemma 2.6.3 *Let G be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected. Then G contains K_4^-, or for any $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, $T \cong K_3$.*

Proof. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, and assume $T \cong K_2$. Then $|S_T| = 5$. Let $a \in N(x) \cap V(A)$. By Lemma 2.6.2, there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $\{a, x\} \subseteq V(T')$. Note that $T' \cong K_2$ and $|S_{T'}| = 5$, or $T' \cong K_3$ and $|S_{T'}| \in \{5, 6\}$. We may assume $|V(A)| \geq 5$; for, if not, then G contains K_4^- by Lemma 2.6.1.

We may assume that if $A \cap C \neq \emptyset$ then $|(S_{T'} \cup S_T) - V(B \cup D)| \geq |S_{T'}| + 1$. For, suppose $A \cap C \neq \emptyset$ and $|(S_{T'} \cup S_T) - V(B \cup D)| \leq |S_{T'}|$. If $|V(A \cap C)| \geq 2$ or $T' \cong K_3$ then $(T', (S_{T'} \cup S_T) - V(B \cup D), A \cap C, B \cup D) \in \mathcal{Q}_x$ and $|V(A \cap C)| \leq |V(A) - \{a\}| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. So assume $|V(A \cap C)| = 1$ and $T' \cong K_2$. Then $|(S_{T'} \cup S_T) - V(B \cup D)| = |S_{T'}| = 5$ and $|V(C)| \geq 2 \leq |V(D)|$. Assume for the moment $A \cap D = \emptyset$. By Lemma 2.6.1, we may assume $|S_{T'} \cap V(A)| = 4$ (as $|S_{T'}| = 5$ and $|V(A)| \geq 5$); so $|S_{T'} \cap V(B)| = 0$, $|S_T \cap V(C)| = 0$, and $|S_{T'} \cap S_T| = 1$. Since $|V(C)| \geq 2$, $B \cap C \neq \emptyset$. So $S_T \cap S_T'$ is a 1-cut in G, contradicting the assumption that G is 5-connected. Hence, $A \cap D \neq \emptyset$. We may assume $|V(A \cap D)| \geq 2$; as otherwise, since G is 5-connected, $G[(A \cap C) \cup (A \cap D) \cup \{a, x\}] \cong K_4^-$. Then $|(S_{T'} \cup S_T) -$

$V(B \cup C)| \geq |S_{T'}| + 1$; otherwise, $(T', (S_{T'} \cup S_T) - V(B \cup C), A \cap D, B \cup C) \in \mathcal{Q}_x$ and $2 \leq |V(A \cap D)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, $|(S_{T'} \cup S_T) - V(A \cup D)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup C)| \leq 4$. Since G is 5-connected, $B \cap C = \emptyset$. Since $|(S_{T'} \cup S_T) - V(B \cup D)| = 5$, $|S_T \cap V(C)| \leq 3$. Therefore, $|V(C)| \leq 4 < |V(A)|$, a contradiction.

Similarly, we may assume that if $A \cap D \neq \emptyset$ then $|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1$.

Suppose $A \cap C = A \cap D = \emptyset$. Then, since $|V(A)| \geq 5$ and $|S_{T'}| \leq 6$, $|S_{T'} \cap V(A)| = |V(A)| = 5$, $|S_T \cap S_{T'}| = 1$, and $|S_{T'} \cap V(B)| = 0$. Since $|S_T| = 5$ and G is 5-connected, we see that $B \cap C = \emptyset$ or $B \cap D = \emptyset$. However, this implies $|V(C)| \leq 4$ or $|V(D)| \leq 4$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

We may thus assume $A \cap C \neq \emptyset$. Then $|(S_{T'} \cup S_T) - V(B \cup D)| \geq |S_{T'}| + 1$. So $|(S_{T'} \cup S_T) - V(A \cup C)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup D)| \leq 4$. Since G is 5-connected, $B \cap D = \emptyset$. In addition, $A \cap D \neq \emptyset$; as otherwise, $|V(D)| \leq 4 < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Therefore, $|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1$. Hence, $|(S_{T'} \cup S_T) - V(A \cup D)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup C)| \leq 4$. Since G is 5-connected, $B \cap C = \emptyset$. Thus, $|V(B)| \leq |S_{T'} - V(T')| = 4$, contradicting the fact $|V(A)| \geq 5$ and $|V(A)|$ is minimum. ∎

The next lemma will allow us to assume that if $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$ then $T \cong K_3$ and $T' \cong K_3$.

Lemma 2.6.4 *Let G be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $T' \cong K_2$. Then one of the following holds:*

(i) *G contains a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(*iii*) *There exist distinct* $x_1, x_2, x_3 \in N(x)$ *such that for any distinct* $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ *contains* TK_5.

Proof. By Lemma 2.6.3, we may assume $T \cong K_3$. By Lemma 2.4.6, we may further assume $|S_T| = 6$. Note the symmetry between C and D, and assume that $V(T) \subseteq S_T - V(D)$. Since $|V(T')| = 2$, $|S_{T'}| = 5$.

Suppose $A \cap C \neq \emptyset$. Then $|(S_{T'} \cup S_T) - V(B \cup D)| \geq 7$; otherwise, $(T, (S_{T'} \cup S_T) - V(B \cup D), A \cap C, B \cup D) \in \mathcal{Q}_x$ and $0 < |V(A \cap C)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, $|(S_{T'} \cup S_T) - V(A \cup C)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup D)| \leq 4$. Since G is 5-connected, $B \cap D = \emptyset$. We may assume $A \cap D \neq \emptyset$; otherwise, $|V(D)| \leq 4$ and, by Lemma 2.6.1, (ii) holds. We may also assume $|V(D)| > |V(A)|$; otherwise, $(T', S_{T'}, D, C) \in \mathcal{Q}_x$ and, by Lemma 2.6.3, G contains K_4^-. Hence, $|V(D) \cap S_T| > |V(A \cap C)| + |V(A) \cap S_{T'}| \geq |V(A) \cap S_{T'}| + 1$. Then, since $|S_T| = 6$ and $V(T) \subseteq S_T - V(D)$, $|V(D) \cap S_T| = 3$ and $|V(A) \cap S_{T'}| = 1$. Hence, $|(S_{T'} \cup S_T) - V(B \cup D)| \leq 4$, a contradiction as G is 5-connected.

Now assume $A \cap C = \emptyset$. Then, since $|S_{T'} \cap V(A)| \leq 4$, we may assume $A \cap D \neq \emptyset$ by Lemma 2.6.1.

Suppose $|(S_{T'} \cup S_T) - V(B \cup C)| = 5$. Then, since $|V(A \cap D)| < |V(A)|$, $|V(A \cap D)| = 1$; otherwise, $(T', (S_{T'} \cup S_T) - V(B \cup C), A \cap D, B \cup C)$ contradicts the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence by Lemma 2.6.1, we may assume $|V(A) \cap S_{T'}| = 4$; so $V(B) \cap S_{T'} = V(D) \cap S_T = \emptyset$. Since G is 5-connected, $B \cap D = \emptyset$. So $|V(D)| = 1$, a contradiction.

Hence, we may assume $|(S_{T'} \cup S_T) - V(B \cup C)| \geq 6$. Then $S_T \cap V(D) \neq \emptyset$ because $|S_{T'}| = 5$. By Lemma 2.6.1, we may assume $B \cap C \neq \emptyset$ (otherwise $|V(C)| \leq 4$). Hence, since G is 5-connected, $|(S_{T'} \cup S_T) - V(A \cup D)| \geq 5$. Since $|(S_{T'} \cup S_T) - V(A \cup D)| + |(S_{T'} \cup S_T) - V(B \cup C)| = |S_T| + |S_{T'}| = 11$, $|(S_{T'} \cup S_T) - V(A \cup D)| = 5$. If $|V(B \cap C)| = 1$ then, since G is 5-connected, $G[T \cup (B \cap C)] \cong K_4^-$. If $|V(B \cap C)| \geq 2$ then, since $V(T) \subseteq (S_{T'} \cup S_T) - V(A \cup D)$, the assertion follows from Lemma 2.4.6. ∎

The proofs of the remaining two results in this section use Lemmas 2.5.1, 2.5.2 and 2.5.3. The following result will allow us to assume that if $(T, S_T, A, B) \in \mathcal{Q}_x$ is chosen to minimize $|V(A)|$ then $N(x) \cap V(A) \neq \emptyset$, which in turn will allow us to choose another quadruple at x.

Lemma 2.6.5 *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ minimizing $|V(A)|$. Then $N(x) \cap V(A) \neq \emptyset$, or one of the following holds:*

(i) *G contains a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(iii) *There exist distinct $x_1, x_2, x_3 \in N(x)$ such that for any distinct $u_1, u_2 \in N(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$ contains TK_5.*

Proof. Suppose $N(x) \cap V(A) = \emptyset$. Then, since G is 5-connected, $|S_T| = 6$ and $T \cong K_3$. Let $V(T) = \{x, x_1, x_2\}$ and $S_T = \{x, x_1, x_2, v_1, v_2, v_3\}$. By Lemma 2.4.8, we may assume $N(x_1) \subseteq V(A) \cup \{x, x_2\}$, and any three independent paths in $G_A := G[A + (S_T - \{x\})] - E(S_T)$ from $\{x_1, x_2\}$ to v_1, v_2, v_3, respectively, with two from x_1 and one from x_2, must include a path from x_2 to v_1.

We wish to apply Lemma 2.5.1. Let G'_A be obtained from G_A by adding a new vertex x'_1 and joining x'_1 to each vertex in $N(x_1) \cap V(G_A)$ with an edge. Thus, in G'_A, x_1 and x'_1 have the same set of neighbors. Note that $\{x_1, x'_1, x_2\}$ and $\{v_1, v_2, v_3\}$ are independent sets in G'_A.

Claim 1. There is no separation (A_1, A_2) in G'_A such that $|V(A_1 \cap A_2)| \leq 3$, $\{x_1, x'_1, x_2\} \subseteq V(A_1)$ and $\{v_1, v_2, v_3\} \subseteq V(A_2)$.

For, suppose such (A_1, A_2) does exist. Then $\{x_1, x'_1\} \not\subseteq V(A_1 \cap A_2)$; for, otherwise, $A_1 - \{x_1, x'_1, x_2\} \neq \emptyset$ (as $\{x_1, x'_1, x_2\}$ is independent in G'_A and x_2 has a neighbor in $V(A)$) and, hence, $(V(A_1 \cap A_2) - \{x'_1\}) \cup \{x, x_2\}$ is a cut in G of size at most 4, a contradiction.

Thus, we may assume by symmetry that $x_1 \notin V(A_1 \cap A_2)$. Then (A_1, A_2) may be chosen so that $x_1' \notin V(A_1 \cap A_2)$ (as x_1' has the same set of neighbors as x_1 in G_A'). Moreover, $V(A_1) - V(A_2) \subseteq \{x_1, x_1', x_2\}$; otherwise $S_T' := V(A_1 \cap A_2) \cup V(T)$ is a cut in G with $|S_T'| \leq 6$, and $G - S_T'$ has a component strictly contained in A, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Since G is 5-connected and $N(x_1) \subseteq V(A) \cup \{x, x_2\}$, $V(A_1 \cap A_2) \cup \{x, x_2\}$ is not a 4-cut in G. So $x_2 \in V(A_1) - V(A_2)$ and $|V(A_1 \cap A_2)| = 3$. Since G is 5-connected and $V(A_1) - V(A_2) \subseteq \{x_1, x_1', x_2\}$, $N(x_1) = \{x, x_2\} \cup V(A_1 \cap A_2)$. Since $N(x_2) \cap V(A_1) \neq \emptyset$, there exists $v \in V(A_1 \cap A_2)$ such that $vx_2 \in E(G)$. Now $G[\{v, x, x_1, x_2\}] \cong K_4^-$ and (ii) holds. This completes the proof of Claim 1.

Since any three disjoint paths in G_A' from $\{x_1, x_2, x_1'\}$ to $\{v_1, v_2, v_3\}$ contains a path from x_2 to v_1, it follows from Claim 1 and Lemma 2.5.1 that G_A' has a separation (J, L) such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, $(L, (x_1, x_2, x_1'), (v_2, v_1, v_3))$ is a ladder along some sequence $b_0 \dots b_m$, where $b_0 = x_2$, $b_m = v_1$, and $w_0 \dots w_n$ is the reduced sequence of $b_0 \dots b_m$. (Note that if (ii) of Lemma 2.5.1 holds then, by Claim 1, $(G_A', (x_1, x_2, x_1'), (v_2, v_1, v_3))$ is a rung, and we let $L = G_A'$ and J consist of v_1 and x_2.)

Since L is a ladder, L contains three disjoint paths P_1, P_2, P_3 from x_1, x_2, x_1', respectively, to $\{v_1, v_2, v_3\}$, with $v_1 \in V(P_2)$. Without loss of generality, we may further assume that $v_2 \in V(P_1)$ and $v_3 \in V(P_3)$. Let $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$, $i \in [m]$, be the rungs in L, with $a_i \in V(P_1)$, $b_i \in V(P_2)$ and $c_i \in V(P_3)$ for $i = 0, \dots, m$. Since G is 5-connected, $(J, w_0, \dots, w_n)$ is planar and, by Lemmas 2.5.2 and 2.5.3, we may assume that the rungs in L have the simple structures as in Lemma 2.5.3.

Claim 2. There exist $t \in V(A)$ and independent paths Q_1, Q_2, Q_3, Q_4, Q_5 in G_A such that Q_1, Q_2, Q_3, Q_4 are from t to x_1, x_2, v_1, v_2, respectively, and Q_5 is from x_1 to v_3; and there exist $t \in V(A)$ and independent paths $Q_1', Q_2', Q_3', Q_4', Q_5'$ in G_A such that Q_1', Q_2', Q_3', Q_4' are from t to x_1, x_2, v_1, v_3, respectively, and Q_5' is from x_1 to v_2.

We may assume that for $i \in [m]$, $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$ is not of type (iv)

as in Lemma 2.5.3. For, suppose $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$ is of type (iv) for some $i \in [m]$, and let $v \in V(R_i) - (\{a_{i-1}, b_{i-1}, c_{i-1}\} \cup \{a_i, b_i, c_i\})$. Then Claim 2 holds with $v, va_{i-1} \cup a_{i-1}P_1x_1, vb_{i-1} \cup b_{i-1}P_2x_2, vb_i \cup b_iP_2v_1, va_i \cup a_iP_1v_2, P_3$ as $t, Q_1, Q_2, Q_3, Q_4, Q_5$, respectively, and with $v, vc_{i-1} \cup c_{i-1}P_3x_1, vb_{i-1} \cup b_{i-1}P_2x_2, vb_i \cup b_iP_2v_1, vc_i \cup c_iP_3v_3, P_1$ as $t, Q_1', Q_2', Q_3', Q_4', Q_5'$, respectively.

We claim that there exists $q \in [m]$, such that $x_1b_q \in E(G)$. Let $q \geq 1$ be the smallest integer such that $(R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))$ is not of type (ii) as in Lemma 2.5.3, which must exist as $x_1 \notin \{v_1, v_2, v_3\}$. Then $a_{q-1} = x_1$ and $c_{q-1} = x_1'$. Since G is 5-connected, $(R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))$ cannot be of type (iii) (thus, must be of type (i)) as in Lemma 2.5.3. Since x_1 and x_1' have the same set of neighbors in G_A', $a_q \neq x_1$ and $c_q \neq x_1'$. Since G is 5-connected, $V(R_q) = \{x_1, x_1', a_q, b_q, c_q\}$. Since $N(x_1) \subseteq V(A) \cup \{x, x_2\}$ and G is 5-connected, $x_1b_q \in E(G)$.

We choose such q to be maximum. Note that $q \neq 0$ as $x_1b_0 \notin E(G_A')$. We now show the existence of t and $Q_i, i \in [5]$; the proof of the existence of t and $Q_i', i \in [5]$, is symmetric (by switching the roles of v_2, P_1 and v_3, P_3).

We may assume that for any choice of P_1, P_3 there does not exist r, with $q < r \leq m$, such that L has disjoint paths S, S' from b_r, x_1 to v_2, v_3, respectively, and internally disjoint from $J \cup P_2$. For, suppose for some choice of P_1, P_3 such r, S, S' exist. By Claim 1, $J \cup P_2$ is 2-connected. So let P_2' denote the path between x_2 and v_1 in $J \cup P_2$ such that the cycle $P_2' \cup P_2$ bounds the infinite face of $J \cup P_2$. Let $t \in V(P_2')$ such that $x_2t \in E(P_2')$. If there exist independent paths L_1, L_2 in $J \cup P_2$ from t to b_q, b_r, respectively, and internally disjoint from P_2', then $L_1 \cup b_qx_1, L_2 \cup S, tx_2, tP_2'v_1, S'$ give the desired Q_1, Q_2, Q_3, Q_4, Q_5, respectively. Thus we may assume that such L_1, L_2 do not exist. So $J \cup P_2$ has a separation (J_1, J_2) such that $|V(J_1 \cap J_2)| \leq 3$, $t \in V(J_1) - V(J_2)$, and $\{b_q, b_r, v_1, x_2\} \subseteq V(J_2)$. By planarity of $J \cup P_2$, $V(J_1 \cap J_2)$ contains x_2 and a vertex $t' \in V(tP_2'v_1)$. Since $V(J_1 \cap J_2)$ cannot be a cut in G, we must have $|V(J_1 \cap J_2)| = 3$, $t' = v_1$, and $V(J_1 \cap J_2) - \{t', x_2\} \subseteq V(b_rP_2v_1)$. Let $b_s \in V(J_1 \cap J_2) - \{t', x_2\}$. Then $V(T) \cup \{a_s, b_s, c_s\}$ is a cut in G separating

$\bigcup_{i=1}^{s} R_s$ from $B+t$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Hence, for any $j > q$, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ must be of type (i) or (ii) as in Lemma 2.5.3 and there is no edge in G'_A from P_2 to $P_1 - x_1$. Also notice that, for $j \leq q$ with $b_{j-1} \neq b_q$, because of edges $x_1 b_q, x'_1 b_q$ in G'_A, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ must be of type (ii) as in Lemma 2.5.3. For $j \leq q$ with $b_{j-1} = b_q$, we see that $V(R_j) = \{x_1, x'_1, a_j, b_q, c_j\}$ as G is 5-connected, and we may assume that $b_q a_j \notin E(G)$ (otherwise, $b_q, b_q x_1, b_q P_2 x_2, b_q P_q v_1, b_q a_q \cup a_q P_1 v_2, P_3$ give the desired $t, Q_1, Q_2, Q_3, Q_4, Q_5$).

Thus, we may assume that for some $j > q$, $\{a_{j-1}, c_{j-1}\} \cap \{a_j, c_j\} = \emptyset$. For, otherwise, $(G_A, x_1, x_2, v_1, v_2, v_3)$ is planar, and the assertion follows from Lemma 2.4.5.

If $R_j - a_{j-1}$ contains disjoint paths S_1, S_2 from b_j, c_{j-1} to a_j, c_j, respectively, then b_j and the paths $S_1 \cup a_j P_1 v_2, x_1 P_3 c_{j-1} \cup S_2 \cup c_j P_3 v_3$ contradict the nonexistence of b_r, S, S'. So assume S_1, S_2 do not exist. Then by Lemma 2.4.10, $(R_j - a_{j-1}, a_j, c_j, b_j, c_{j-1})$ is planar. By Lemma 2.4.5, we may assume $|V(R_j - a_{j-1})| \leq 5$.

If $|V(R_j - a_{j-1})| = 5$ then there exists $v \in V(R_j) - \{a_{j-1}, a_j, b_j, c_{j-1}, c_j\}$ such that v is adjacent to all of $\{a_{j-1}, a_j, b_j, c_{j-1}, c_j\}$; so b_j and the paths $b_j v a_j \cup a_j P_1 v_2, P_3$ contradict the nonexistence of b_r, S, S'.

Hence, we may assume $|V(R_j - a_{j-1})| = 4$. Then, since R_j has no cut of size at most 3 separating $\{a_{j-1}, b_{j-1}, c_{j-1}\}$ from $\{a_j, b_j, c_j\}$, we must have $a_{j-1} c_j, a_j c_{j-1} \in E(G)$. Note that there exists $t > q$ such that L has a path Z from b_t to $z \in V(x_1 P_1 a_{j-1} - x_1) \cup V(x'_1 P_3 c_{j-1} - x'_1)$ and internally disjoint from $J \cup P_1 \cup P_2 \cup P_3$; for otherwise, $\{a_j, b_j, c_j, x_1\}$ would be a cut in G. If $z \in V(x_1 P_1 a_{j-1} - x_1)$ then b_t and the paths $Z \cup z P_1 v_2, P_3$ contradict the nonexistence of b_r, S, S'. So assume $z \in V(x_1 P_3 c_{j-1} - x_1)$. Then b_t and the paths $Z \cup z P_3 c_{j-1} \cup c_{j-1} a_j \cup a_j P_1 v_2, x_1 P_1 a_{j-1} \cup a_{j-1} c_j \cup c_j P_3 v_3$ contradict the nonexistence of b_r, S, S', with $x'_1 P_3 c_{j-1} \cup c_{j-1} a_j \cup a_j P_1 v_2, x_1 P_1 a_{j-1} \cup a_{j-1} c_j \cup c_j P_3 v_3$ as P_1, P_3, respectively. This completes the proof of Claim 2.

Now that we have the paths in Claim 2, we turn to $G_B := G[B + S_T - x_1]$. Choose $x_3 \in N(x) \cap V(B)$, let $u_1 := x_3$ and let $u_2 \in N(x) - \{x_1, x_2, x_3\}$ be arbitrary. Note that

$u_2 \in S_T \cup V(B)$. We wish to prove (iii) by attempting to find a TK_5 in $G' := G - \{xv : v \notin \{u_1, u_2, x_1, x_2\}\}$. Since G is 5-connected and $N(x_1) \cap V(B) = \emptyset$, G_B has four independent paths B_1, B_2, B_3, B_4 from u_1 to v_1, v_2, v_3, x_2, respectively, and we may assume that these paths are induced.

Claim 3. We may assume $u_2 \notin S_T$.

For, suppose $u_2 \in S_T$. If $u_2 = v_1$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup v_1x) \cup u_1x \cup B_4 \cup (B_2 \cup Q_4) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. If $u_2 = v_2$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2x) \cup u_1x \cup B_4 \cup (B_1 \cup Q_3) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. Now assume $u_2 = v_3$. Then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup v_3x) \cup u_1x \cup B_4 \cup (B_1 \cup Q'_3) \cup (B_2 \cup Q'_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. This completes the proof of Claim 3.

Let P be a path in G_B from u_2 to some $w_2 \in V(B_1 \cup B_2 \cup B_3 \cup B_4) - \{u_1\}$ and internally disjoint from $B_1 \cup B_2 \cup B_3 \cup B_4$.

Claim 4. We may assume that for any choice of P, $w_2 \in V(B_4)$.

For, if $w_2 \in V(B_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup v_1B_1w_2 \cup P \cup u_2x) \cup u_1x \cup B_4 \cup (B_2 \cup Q_4) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. If $w_2 \in V(B_2)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2B_2w_2 \cup P \cup u_2x) \cup u_1x \cup B_4 \cup (B_1 \cup Q_3) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. If $w_2 \in V(B_3)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup v_3B_3w_2 \cup P \cup u_2x) \cup u_1x \cup B_4 \cup (B_1 \cup Q'_3) \cup (B_2 \cup Q'_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. This completes the proof of Claim 4.

Let U_2 denote the $(B_1 \cup B_2 \cup B_3)$-bridge of G_B containing $B_4 + u_2$. That is, U_2 is the subgraph of G_B induced by the edges in the component of $G_B - (B_1 \cup B_2 \cup B_3)$ containing $B_4 + u_2$ and the edges from that component to $B_1 \cup B_2 \cup B_3$.

Claim 5. We may assume that $V(U_2) \cap V(B_2 \cup B_3) = \{u_1\}$.

For, suppose there exists $w \in V(U_2) \cap V(B_2 \cup B_3)$ such that $w \neq u_1$. By symmetry, we may assume $w \in V(B_2 - u_1)$ and choose w so that wB_2v_2 is minimal.

Then U_2 has a path X between x_2 to w and internally disjoint from $B_1 \cup B_2 \cup B_3$, and a path from u_2 to some $u_2' \in V(X)$ and internally disjoint from $X \cup B_1 \cup B_2 \cup B_3$. Since G is 5-connected, U_2 has four independent paths from u_2' to four distinct vertices in $V(U_2) \cap V(B_1 \cup B_2 \cup B_3)$ and internally disjoint from $B_1 \cup B_2 \cup B_3$. Thus, by Lemma 2.4.11, U_2 contains independent paths L_1, L_2, L_3, L_4 from u_2' to u_2, x_2, w, w', respectively, and internally disjoint from $B_1 \cup B_2 \cup B_3$, where $w' \in V(B_1 \cup B_2 \cup B_3)$.

If $w' \in V(wB_2u_1 - w)$ then $T \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup P_1) \cup (u_1B_2w' \cup L_4) \cup u_1x \cup (B_1 \cup P_2) \cup (B_3 \cup P_3)$ is a TK_5 in G' with branch vertices u_1, u_2', x, x_1, x_2. (Note we identify x_1' with x_1 when we use P_3.)

If $w' \in V(B_1 - u_1)$ then $T \cup Q_1' \cup Q_2' \cup (Q_4' \cup B_3 \cup u_1x) \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup Q_5') \cup (L_4 \cup w'B_1v_1 \cup Q_3')$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2.

If $w' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup B_1 \cup u_1x) \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup Q_4) \cup (L_4 \cup w'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2. This completes the proof of Claim 5.

Now let $z \in V(B_1 \cap U_2)$ such that zB_1v_1 is minimal. Since G is 5-connected, there exists a path Y in $G_B - x$ from some $y \in V(zB_1u_1) - \{u_1, z\}$ to some $y' \in V(B_2 \cup B_3) - \{u_1\}$ and internally disjoint from $U_2 \cup B_1 \cup B_2 \cup B_3$.

Claim 6. We may assume that $G[U_2 - B_1 + z]$ has no independent paths from u_2 to x_2, z, respectively.

For, suppose $G[U_2 - B_1 + z]$ (and hence $G[U_2 \cup zB_1u_1]$) has independent paths from u_2 to x_2, z, respectively. Then by Lemma 2.4.11, $G[U_2 \cup zB_1u_1]$ has independent paths L_1, L_2, L_3, L_4 from u_2 to distinct vertices x_2, z, z_1, z_2, respectively, and internally disjoint from B_1, where u_1, z_2, z_1, z occur on B_1 in the order listed. Possibly, $u_1 = z_2$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q_1' \cup Q_2' \cup (Q_4' \cup B_3 \cup u_1x) \cup u_2x \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q_3') \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q_5')$ is a TK_5 in G' with branch vertices t, u_2, x, x_1, x_2.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup u_2x \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_2, x, x_1, x_2.

By Claim 6, $G[U_2 - B_1 + z]$ has a 1-separation (U_{21}, U_{22}) such that $u_2 \in V(U_{21}) - V(U_{22})$ and $\{x_2, z\} \subseteq V(U_{22})$. We choose this separation so that U_{22} is minimal. Let u_2' denote the unique vertex in $V(U_{21} \cap U_{22})$. By the minimality of U_{22}, we see that U_{22} has independent paths L_1, L_2 from u_2' to x_2, z, respectively.

Claim 7. We may assume that u_2' has exactly two neighbors in U_{22}.

For, otherwise, by the minimality of U_{22}, $G[U_{22} \cup zB_1u_1] - u_1$ has three independent paths from u_2' to three distinct vertices in $V(zB_1u_1 - u_1) \cup \{x_2\}$. So by Lemma 2.4.11, $G[U_{22} \cup zB_1u_1] - u_1$ has independent paths L_1', L_2', L_3' from u_2' to x_2, z, z_1, respectively, and internally disjoint from B_1, where z, z_1, u_1 occur on B_1 in order. Let L be a path in U_{21} from u_2 to u_2'.

If $y' \in V(B_2 - u_1)$ then $T \cup Q_1' \cup Q_2' \cup (Q_4' \cup B_3 \cup u_1x) \cup (L \cup u_2x) \cup L_1' \cup (L_2' \cup zB_2v_1 \cup Q_3') \cup (L_3' \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q_5')$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup (L \cup u_2x) \cup L_1' \cup (L_2' \cup zB_2v_1 \cup Q_3) \cup (L_3' \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2. This completes the proof of Claim 7.

Since G is 5-connected, it follows from Claim 7 that u_2' has at least two neighbors in U_{21}. Since all paths from u_2 to $B_1 \cup B_2 \cup B_3 \cup B_4$ must end on B_4, $G[U_{21} \cup zB_1u_1] - \{z, u_1\}$ has independent paths L_3, L_4 from u_2' to z_1, u_2, respectively, and internally disjoint from B_1, where $z_1 \in V(zB_1u_1) - \{z, u_1\}$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q_1' \cup Q_2' \cup (Q_4' \cup B_3 \cup u_1x) \cup (L_4 \cup u_2x) \cup L_1 \cup (L_2 \cup zB_2v_1 \cup Q_3') \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q_5')$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup (L_4 \cup u_2x) \cup L_1 \cup (L_2 \cup zB_2v_1 \cup Q_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2.∎

We conclude this section with another technical lemma, which deals with a special case that occurs in the proof of Lemma 2.7.5. It is included in this section because its proof also makes use of Lemmas 2.5.1, 2.5.2 and 2.5.3.

Lemma 2.6.6 *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ such that $|V(A)|$ is minimum, and suppose there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $T' \cong K_3$, $T' \cap A \neq \emptyset$, $V(A \cap C) = S_T \cap V(C) = V(B \cap D) = V(B) \cap S_{T'} = \emptyset$, $|V(A) \cap S_{T'}| = |V(D) \cap S_T| = |V(D \cap T)| = 1$, and $|S_T \cap S_{T'}| = 5$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, we have G/H is not 5-connected, $|V(H \cap A)| \leq 1$, and $H \cong K_3$ when $H \cap A \neq \emptyset$. Then one of the following holds:*

(i) *G has a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(iii) *There exist $x_1, x_2, x_3 \in N(x)$ such that, for any distinct $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*

Proof. Note that $|S_T| = |S_T \cap S_{T'}| + |V(D \cap T)| = 6$. Let $V(T) = \{x, w, x_1\}$ and $T' = \{x, a, b\}$ such that $V(A) \cap S_{T'} = \{a\}$ and $V(D) \cap S_T = \{w\}$, and let $S_T \cap S_{T'} = \{x, x_1, b, z_1, z_2\}$. Then $|V(D)| = |V(A)| = |V(A \cap D)| + 1$. Moreover,

(1) $|N(s) \cap V(A)| \geq 2$ for $s \in \{b, z_1, z_2\}$,

for, otherwise, $(T, (S_T - \{s\}) \cup (N(s) \cap V(A)), A - N(s), G[B + s]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. We may assume that

(2) G has no edge from $T - x$ to $T' - x$,

as otherwise $G[T \cup T']$ contains K_4^- and (ii) holds. We may also assume

(3) $N(x_1) \cap V(D) \neq \{w\}$ and $N(w) \cap V(A) \neq \emptyset$,

for, otherwise, let $S := S_T \setminus \{x_1\}$ and $B' = G[B + x_1]$ if $N(x_1) \cap V(D) = \{w\}$, and let $S := S_T \setminus \{w\}$ and $B' = G[B + w]$ if $N(w) \cap V(A) = \emptyset$; then $(xw, S, A, B') \in \mathcal{Q}_x$, and (ii) follows from Lemma 2.6.3. We may further assume that

(4) for any $x' \in N(x) \cap V(A \cap D)$, $xx'z_1x$ or $xx'z_2x$ is a triangle.

For, let $x' \in N(x) \cap V(A \cap D)$. By Lemma 2.6.2, we may assume that there exists $H \subseteq G$ with $x, x' \in V(H)$ and $H \cong K_2$ or $H \cong K_3$. By the assumption of this lemma, $H \cong K_3$ and $V(H) \cap S_T \neq \{x\}$. If $V(H) \cap \{b, x_1\} \neq \emptyset$ then $H \cup T$ or $H \cup T'$ contains K_4^-. So we may assume $V(H) \cap \{z_1, z_2\} \neq \emptyset$ and, hence, $xx'z_1x$ or $xx'z_2x$ is a triangle.

We may assume that

(5) $|N(x) \cap V(A \cap D)| \leq 2$.

For, otherwise, by (4), there exist $i \in [2]$ and distinct $x', x'' \in N(x) \cap V(A \cap D) \cap N(z_i)$. So $G[x', x'', x, z_i]$ contains K_4^-, and (ii) holds.

We now distinguish two cases.

Case 1. $z_i \notin N(x)$ for $i \in [2]$.

Then by (4), $N(x) \cap V(A \cap D) = \emptyset$. We prove that (iii) holds with $x_2 = w$ and $x_3 = b$. Let $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$. Since G is 5-connected and $z_1, z_2 \notin N(x)$, we may assume $y_1 \in V(B \cap C)$. Then $G_B := G[B + \{b, x_1, z_1, z_2\}]$ has independent paths Y_1, Y_2, Y_3, Y_4 from y_1 to z_1, z_2, x_1, b, respectively.

We may assume that $wz_i \notin E(G)$ for $i \in [2]$. For, suppose $wz_1 \in E(G)$. If $G[A + \{b, w, x_1\}]$ has independent paths Q_1, Q_2 from b to x_1, w, respectively, then $T \cup bx \cup Q_1 \cup Q_2 \cup y_1x \cup (Y_1 \cup z_1w) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, w, x, x_1, y_1. So we may assume that such Q_1, Q_2 do not exist. Then $G[A + \{b, w, x_1\}]$ has a cut vertex v separating b from $\{w, x_1\}$. Let D denote the component of $G[A + \{b, w, x_1\}] - v$ containing b. Since $|N(b) \cap V(A)| \geq 2$ (by (1)), $|V(D)| \geq 2$. Now $\{b, v, x, z_1, z_2\}$ is a cut in G, and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, v, x, z_1, z_2\}$, $|V(G_1)| \geq 6$ and $\{a, b\} \subseteq V(G_1)$, and $B + \{w, x_1\} \subseteq G_2$. By the choice of (T, S_T, A, B) with $|V(A)|$ minimum, $|V(G_1)| = 6$. Let $u \in V(G_1) - V(G_2)$. If $u = a$ then, $V(G_1 \cap G_2) \subseteq N(a)$ (since G is 5-connected) and $bv \in E(G)$ (since $|N(b) \cap V(A)| \geq 2$); so $G[\{a, b, v, x\}] \cong K_4^-$, and (ii) holds. So assume $u \neq a$. Then $v = a$ and $G[\{b, u, v, x\}]$ contains K_4^-; so (ii) holds.

We may assume that $G_A := G[A+\{b, w, x_1, z_1, z_2\}]$ does not contain three independent paths, with one from x_1 to b, one from b to w, and one from w to z_i for some $i \in [2]$. For, otherwise, such three paths and $T \cup bx \cup y_1x \cup Y_i \cup Y_3 \cup Y_4$ form a TK_5 in G' with branch vertices b, w, x, x_1, y_1.

We wish to apply Lemma 2.5.1. Let G'_A be the graph obtained from G_A by identifying z_1 and z_2 as z', and duplicating w, b with w', b', respectively (adding edges from w' to all vertices in $N(w)$, and from b' to all vertices in $N(b)$). Then any three disjoint paths in G'_A from $\{w, x_1, w'\}$ to $\{b, z', b'\}$, if exist, must contain a path from x_1 to z'.

Suppose G'_A has a separation (A_1, A_2) such that $|V(A_1 \cap A_2)| \leq 2$, $\{w, x_1, w'\} \subseteq V(A_1)$, and $\{b, z', b'\} \subseteq V(A_2)$. Since w and w' have the same set of neighbors in G'_A, we may assume $\{w, w'\} \subseteq V(A_1 \cap A_2)$ or $\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset$. If $\{w, w'\} \subseteq V(A_1 \cap A_2)$ then $V(A_1) = \{x_1\} \cup V(A_1 \cap A_2)$ as $\{x, x_1, w\}$ cannot be a cut in G; hence, $N(x_1) \cap V(D) = \{w\}$, contradicting (3). So $\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset$. Suppose $\{b, b,', z'\} \cap V(A_1 \cap A_2) = \emptyset$. Then, since $wz_i \notin E(G)$ for $i \in [2]$, $V(A_1 \cap A_2) \cup \{x_1, x\}$ is a cut in G separating w from $B+\{b, z_1, z_2\}$, contradicting the fact that G is 5-connected. So $\{b, b,', z'\} \cap V(A_1 \cap A_2) \neq \emptyset$. Note that $\{b, b'\} \not\subseteq V(A_1 \cap A_2)$; as otherwise $\{b, x, x_1\}$ would be a cut in G. Thus, we may assume that $b, b' \notin V(A_1 \cap A_2)$ as b and b' have the same set of neighbors in G'_A. Hence, $z' \in V(A_1 \cap A_2)$. Now $S := \{x, x_1, z_1, z_2\} \cup (V(A_1 \cap A_2) - \{z'\})$ is a cut in G separating w from $B + b$. Since G is 5-connected, $x_1 \notin V(A_1 \cap A_2)$. If $|V(A_1 - x_1 - A_2)| \geq 2$ then $(xx_1, S, A_1 - x_1 - A_2, G - S - A_1) \in \mathcal{Q}_x$ which contradicts the choice of (T, S_T, A, B) with $|V(A)|$ minimum. So $V(A_1 - x_1 - A_2) = \{w\}$. Since G is 5-connected, $wz_i \in E(G)$ for $i \in [2]$, a contradiction.

Hence, by Lemma 2.5.1, G'_A has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is planar (since G is 5-connected), $(L, (w, x_1, w'), (b, z', b'))$ is a ladder along a sequence $b_0 \ldots b_m$, where $b_0 = x_1$, $b_m = z'$, and $w_0 \ldots w_n$ is the reduced sequence of $b_0 \ldots b_m$. Moreover, we may assume that L has disjoint induced paths P_1, P_2, P_3 from w, x_1, w' to b, z', b', respectively, and J is a connected plane graph with P_2 as part of the outer walk of J and $w_0, \dots, w_n$ occurring on P_2 in order. (When (ii) of Lemma 2.5.1 holds, we let $J = P_2$.) Note that by Lemmas 2.5.2 and 2.5.3, each rung of $(L, (w, x_1, w'), (b, z', b'))$ is of type (i)–(iv) as in Lemma 2.5.3, with possible exceptions of those rungs containing z'. Let $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$, $j \in [m]$, be the rungs in $(L, (w, x_1, w'), (b, z', b'))$ such that $a_j \in V(P_1)$ and $c_j \in V(P_3)$ for $j = 0, 1, \dots, m$.

We now show that there exists $t \in N(w)$ such that $t \in V(P_2) - \{x_1, z'\}$. For, suppose such t does not exist. Choose the largest j such that $\{w, w'\} \subseteq V(R_j)$ and $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is not of type (ii) in Lemma 2.5.3, which is well defined as $w \neq b$. Since G is 5-connected and w and w' have the same set of neighbors in G'_A, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ cannot be of type (iii) as in Lemma 2.5.3. Moreover, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is not of type (iv) as in Lemma 2.5.3, as otherwise G contains K_4^- (obtained from $R_j - \{b_{j-1}, b_j\}$ after identifying w with w'). So $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is of type (i) as in Lemma 2.5.3. Now $V(R_j) = \{a_j, b_j, c_j, w, w'\}$, as otherwise $\{a_j, b_j, c_j, w\}$ would be a cut in G. Then $wb_j \in E(G)$; for otherwise, $N(w) \subseteq \{a_j, c_j, x, x_1\}$, a contradiction. Hence $t := b_j$ is as desired.

Without loss of generality, we may assume that the edge of P_2 incident with z' corresponds to the edge of G incident with z_1. We view P_3 as a path in G_A from b to w. Then $G_A - V(P_1 \cup P_3) - z_2$ has independent paths from t to x_1, z_1, respectively. Hence, by Lemma 2.4.11, G_A has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from t to $x_1, w, z_1, (V(P_1 \cup P_3) - \{w\}) \cup \{z_2\}$, respectively, with only t in common, and internally disjoint from $P_1 \cup P_3$. Without loss of generality, we may assume that Q_4 ends at $t' \in V(P_3)$.

If $G_B - x$ contains disjoint paths S_1, S_2 from z_1, b to y_1, x_1, respectively, then $T \cup bx \cup P_1 \cup S_2 \cup Q_1 \cup Q_2 \cup (Q_3 \cup S_1 \cup y_1x) \cup (Q_4 \cup t'P_3b)$ is TK_5 in G' with branch vertices b, t, w, x, x_1. Hence, we may assume such S_1, S_2 do not exist. Then by Lemma 2.4.10, there exists a collection $\mathcal{D}$ of subsets of $(G_B - x) - \{z_1, b, y_1, x_1\}$ such that $(G_B - x, \mathcal{D}, z_1, b, y_1, x_1)$ is 3-planar.

If $(G_B - x, \{b, x_1, z_1, z_2\})$ is planar then the assertion of the lemma follows from Lemma 2.4.5, with the cut $\{b, x, x_1, z_1, z_2\}$ giving the required 5-separation for Lemma 2.4.5.

So we may assume that either $\mathcal{D} = \emptyset$ and z_2 does not belong to the facial walk of $G_B - x$ containing $\{b, x_1, y_1, z_1\}$, or $\mathcal{D} = \{D\}$ for some $D \subseteq V(G_B - x) - \{b, x_1, y_1, z_1\}$ and $z_2 \in D$. Thus, since G is 5-connected and $(G_B - x, \{z_1, b, y_1, x_1\})$ is 3-planar, $G_B - x$ has disjoint paths S_1', S_2' from z_2, b to y_1, x_1, respectively. Moreover, if b has degree at least two in $G_B - x$ then $G_B - x$ has independent paths Y, Y_2', Y_3', Y_4', with Y from b to x_1 and Y_2', Y_3', Y_4' from y_1 to z_2, x_1, b, respectively.

We may assume that $G_A' - J$ contains a path Z from z_2 to some $z_2' \in V(P_1 \cup P_3) - \{b, b'\}$ and internally disjoint from $P_1 \cup P_3$. For, suppose not. Then, since $|N(z_2) \cap V(A)| \geq 2$ (by (1)), z_2 has at least two neighbors in $J - z'$. Then $G_A' - V(P_1 \cup P_3) - z_1$ has independent paths from t to x_1, z_2, respectively; for otherwise, $G_A' - V(P_1 \cup P_3) - z_1$ has a cut vertex $v \in V(tP_2x_2)$ separating t from $\{x_1, z_2\}$ and, hence, $V(T) \cup \{v, z_1, z_2\}$ is a cut in G, contradicting the choice of S_T with $|V(A)|$ minimum. Hence, by Lemma 2.4.11, G_A has five independent paths $Q_1', Q_2', Q_3', Q_4', Q_5'$ from t to $x_1, w, z_2, (V(P_1 \cup P_3) - \{w, b'\}) \cup \{z_1\}$, respectively, with only t in common, and internally disjoint from $P_1 \cup (P_3 - \{b', w'\})$. Without loss of generality, we may assume that Q_4' ends at $t'' \in V(P_3)$. Then $T \cup bx \cup P_1 \cup S_2' \cup Q_1' \cup Q_2' \cup (Q_3' \cup S_1' \cup y_1x) \cup (Q_4' \cup t''P_3b)$ is TK_5 in G' with branch vertices b, t, w, x, x_1.

Without loss of generality, we may assume that $z_2' \in V(P_3)$. We may further assume that b has only one neighbor in $G_B - x$; for, otherwise, $T \cup bx \cup P_1 \cup Y \cup y_1x \cup (Y_2' \cup Z \cup z_2'P_3w) \cup Y_3' \cup Y_4'$ is a TK_5 in G' with branch vertices b, w, x, x_1, y_1.

Thus, since G is 5-connected and $bw \notin E(G)$ (by (2)), b has a neighbor $u \in V(A) - V(P_1 \cup P_3)$. We choose u and the rung $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ such that $b, b', u \in V(R_j)$. Since b and b' have the same set of neighbors in G_A', $a_{j-1} = b$ if, and only if, $c_{j-1} = b'$. Moreover, we must have $b_j = z'$ because of the path Z.

First, suppose $b_{j-1} = z'$. Then $a_{j-1} \neq b$ and $c_{j-1} \neq b'$. If z_2 has no neighbor in $V(G_A' -$

$J - R_j$) then $V(T) \cup \{a_{j-1}, c_{j-1}, z_1\}$ is a cut in G separating $a_{j-1}P_1w \cup c_{j-1}P_3w \cup (J - z')$ from $B \cup (R_j - \{b', z'\})$, contradicting the choice of (T, S_T, A, B) with $|V(A)|$ minimum. Thus, z_2 has a neighbor in $V(G'_A - J - R_j)$; so the above path Z may be chosen to be disjoint from R_j. Let S be a path in $R_j - \{a_{j-1}, c_{j-1}\}$ from b to z_1 (which must exist as otherwise $\{a_{j-1}, c_{j-1}, z_2\} \cup V(T)$ is a cut in G contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum). So $T \cup bx \cup P_1 \cup (S \cup z_1P_2x_1) \cup y_1x \cup (Y_2 \cup Z \cup z'_2P_3w) \cup Y_3 \cup Y_4$ is TK_5 in G' with branch vertices b, w, x, x_1, y_1.

Now assume $b_{j-1} \neq z' = b_j$. Since $b_{j-1} \neq b_j$ and since b and b' have the same set of neighbors in G'_A, we must have $a_{j-1} = b$ and $c_{j-1} = b'$. If $u \in \{b_{j-1}, b_j\}$ then, since $bz' \notin E(G'_A)$, $u = b_{j-1}$; and let $S = bb_{j-1}$. Now suppose $u \notin \{b_{j-1}, b_j\}$. Then $\{b, b_{j-1}, x, z_1, z_2\}$ is a cut in G separating u from $(J - z') \cup B$. By the choice of (T, S_T, A, B) that $|V(A)|$ is minimum, $\{u\} = V(R_j) - \{b, b', b_{j-1}, z'\}$. Since G is 5-connected, $N(u) = \{b, b_{i-1}, x, z_1, z_2\}$. Let $S = bub_{j-1}$. Since $|N(z_2) \cap V(A)| \geq 2$ (by (1)), the path Z may be chosen to be disjoint from R_j. So $T \cup bx \cup P_1 \cup (S \cup b_{j-1}P_2x_1) \cup y_1x \cup (Y_2 \cup Z \cup z'_2P_3w) \cup Y_3 \cup Y_4$ is TK_5 in G' with branch vertices b, w, x, x_1, y_1.

Case 2. $N(x) \cap \{z_1, z_2\} \neq \emptyset$.

Without loss of generality, we may assume $xz_1 \in E(G)$. We may further assume z_1 is not adjacent to any of $\{a, b, w, x_1\}$; for otherwise, $G[T + z_1]$ or $G[T' + z_1]$ contains K_4^-, and (ii) holds. We wish to prove (iii), with $x_2 = b$ and $x_3 = z_1$. Let $y_1, y_2 \in N(x) - \{b, x_1, z_1\}$ be distinct.

Subcase 2.1. There exists some $i \in [2]$ such that $y_i \in V(B) \cup \{z_2\}$.

Without loss of generality, assume $y_1 \in V(B) \cup \{z_2\}$ and, whenever possible, let $y_1 \in V(B)$. Let $G_B := G[B + \{b, x_1, z_1, z_2\}]$. When $y_1 \in V(B)$ let $t = y_1$ and let Y_1, Y_2, Y_3, Y_4, Y_5 be independent paths in $G[B]$ from t to z_1, y_1, b, x_1, z_2, respectively. When $y_1 = z_2$ let $t = y_1$ and let Y_1, Y_2, Y_3, Y_4 be independent paths in $G[B]$ from t to z_1, y_1, b, x_1, z_2, respectively. Let $G_A = G[A + \{b, w, x_1, z_1\}]$.

We may assume that there is no cycle in G_A containing $\{b, x_1, z_1\}$. For, such a cycle and

$xb \cup xx_1 \cup xz_1 \cup Y_1 \cup (Y_2 \cup y_1x) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, t, x, x_1, z_1.

We may also assume that G_A is 2-connected. To see this, we first assume $N(x_1) \cap N(w) = \{x\}$; for otherwise, letting $u \in (N(x_1) \cap N(w)) - \{x\}$ we see that $G[T + u]$ contains K_4^- and (ii) holds. Therefore, since $N(w) \cap V(A) \neq \emptyset \neq N(x_1) \cap V(A)$ (by (3)), it suffices to show that $G[A + \{b, z_1\}]$ is 2-connected. So assume for a contradiction that there exists a separation (A_1, A_2) in $G[A + \{b, z_1\}]$ such that $|V(A_1 \cap A_2)| \leq 1$. Without loss of generality, let $|\{b, z_1\} \cap V(A_1)| \leq 1$. Then $V(A_1) \not\subseteq V(A_2) \cup \{b, z_1\}$ as $|N(s) \cap V(A)| \geq 2$ for $s \in \{b, z_1\}$ (by (1)). Hence, $V(T) \cup (\{b, z_1\} \cap A_1) \cup V(A_1 \cap A_2) \cup \{z_2\}$ is a cut in G of size at most 6 which separates A_1 from the rest of G, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Then, since G_A has no cycle containing $\{b, x_1, z_1\}$, (i), or (ii), or (iii) of Lemma 2.4.12 holds for G_A and $\{b, x_1, z_1\}$. So for each $u \in \{b, x_1, z_1\}$, G_A has a 2-cut S_u separating u from $\{b, x_1, z_1\} - \{u\}$, and let D_u denote a union of components of $G_A - S_u$ such that $u \in V(D_u)$ for $u \in \{b, x_1, z_1\}$ and D_b, D_{x_1}, D_{z_1} are pairwise disjoint. We choose S_u and D_u, $u \in \{b, x_1, z_1\}$, to maximize $D_b \cup D_{x_1} \cup D_{z_1}$. Note that, since $wx_1 \in E(G)$, $w \notin V(D_b \cup D_{z_1})$.

We claim that for $u \in \{b, x_1, z_1\}$, $V(D_u) = \{u\}$. For, otherwise, $S := S_u \cup \{u, x, z_2\}$ is a cut in G separating $D_u - u$ from the rest of G. If $|V(D_u)| \geq 3$ then $(ux, S, D_u, G - S - D_u) \in \mathcal{Q}_x$ contradicts the choice of (T, S_T, A, B) with $|V(A)|$ minimum. So let $V(D_u) = \{u, u'\}$ and let $S_u = \{s_u, t_u\}$. Since G is 5-connected, $N(u') = \{s_u, t_u, u, x, z_2\}$. Since $|N(u) \cap V(A + w)| \geq 2$ (by (1) and (3)), we may assume that $us_u \in E(G)$. Then $G[\{s_u, u, u', x\}]$ contains K_4^-, and (ii) holds.

For $u \in \{b, x_1, z_1\}$, let $S_u = \{s_u, t_u\}$. Since G_A is 2-connected, $\{us_u, ut_u\} \subseteq E(G)$. Note $a \in \{s_b, t_b\}$; so we may assume $s_bt_b \notin E(G)$ because otherwise $G[\{x, b, s_b, t_b\}]$ contains K_4^-, and (ii) holds. Similarly, $w \in \{s_{x_1}, t_{x_1}\}$ and we may assume $s_{x_1}t_{x_1} \notin E(G)$. If (i) of Lemma 2.4.12 occurs then $ax_1 \in E(G)$, contradicting (2). If (iii) of Lemma 2.4.12 occurs then let R_1, R_2 be the components of $G_A - V(D_b \cup D_{x_1} \cup D_{z_1})$ and assume without loss of generality that $s_u \in V(R_1)$ and $t_u \in V(R_2)$ for $u \in \{b, x_1, z_1\}$. By symmetry, assume $w \notin V(R_1)$. Hence, $(xb, \{x, b, x_1, s_{z_1}, z_2\}, R_1 - s_{z_1}, G - R_1 - \{x, b, x_1, z_2\}]) \in \mathcal{Q}_x$ with $2 \leq |V(R_1 - s_{z_1})| < |V(A)|$, contradicting the choice of (T, S_T, A, B).

So we may assume that (ii) of Lemma 2.4.12 holds. Without loss of generality let R_1, R_2 be the components of $G - V(D_b \cup D_{x_1} \cup D_{z_1})$ containing $z = s_b = s_{x_1} = s_{z_1}$, $\{t_b, t_{x_1}, t_{z_1}\}$, respectively. By (2), $z \neq a$ and $z \neq w$. So $a = t_b$ and $w = t_{x_1}$. Thus, we may assume $xz \notin E(G)$ as, otherwise, $G[T + z]$ contains K_4^- and (ii) holds. Hence, $R_1 = R_2$ (otherwise z would have degree at most 4 in G). By (1) and by the maximality of $D_b \cup D_{x_1} \cup D_{z_1}$, $G[R_2 + z_2]$ is 2-connected (since G is 5-connected).

We claim that there exist distinct $t_1, t_2 \in \{a, w, t_{z_1}\}$ such that $G[R_2 + z_2]$ contains disjoint paths P_1, P_2 from z, t_1 to z_2, t_2, respectively. For, suppose $\{a, w\}$ cannot serve as $\{t_1, t_2\}$. Then, by Lemma 2.4.10, $(G[R_2 + z_2], a, z_2, w, z)$ is 3-planar. Hence, $G[R_2 + z_2]$ has disjoint paths from z, a to z_2, t_{z_1}, respectively, or disjoint paths from z, w to z_2, t_{z_1}, respectively.

Suppose $z_2 \neq y_1$. Recall the definition of t and the paths Y_1, Y_2, Y_3, Y_4, Y_5. If $\{t_1, t_2\} = \{a, w\}$ then $bxx_1zb \cup xz_1z \cup (x_1w \cup P_2 \cup ab) \cup (Y_2 \cup y_1x) \cup (Y_5 \cup P_1) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, t, x, x_1, z. If $\{t_1, t_2\} = \{a, t_{z_1}\}$ then $bxz_1zb \cup xx_1z \cup (z_1t_{z_1} \cup P_2 \cup ab) \cup Y_1 \cup (Y_2 \cup y_1x) \cup Y_3 \cup (Y_5 \cup P_1)$ is a TK_5 in G' with branch vertices b, t, x, z, z_1. If $\{t_1, t_2\} = \{w, t_{z_1}\}$ then $x_1xz_1zx_1 \cup xbz \cup (x_1w \cup P_2 \cup t_{z_1}z_1) \cup Y_1 \cup (Y_2 \cup y_1x) \cup Y_4 \cup (Y_5 \cup P_1)$ is a TK_5 in G' with branch vertices t, x, x_1, z, z_1.

So assume $z_2 = y_1$. Then $y_2 \neq z_2$; and hence, by the choice of y_1, we have $y_2 \in V(A) \cup \{w\}$. If $R_2 - z$ has independent paths S_1, S_2, S_3 from y_2 to a, w, t_{z_1}, respectively, then $xbzx_1x \cup y_2x \cup (S_1 \cup ab) \cup (S_2 \cup wx_1) \cup Y_3 \cup Y_4 \cup (Y_1 \cup z_1t_{z_1} \cup S_3) \cup (Y_2 \cup z_2x)$ is a TK_5 in G' with branch vertices b, t, x, x_1, y_2. So assume such S_1, S_2, S_3 do not exist. Then R_2 has a separation (A_1, A_2) such that $z \in V(A_1 \cap A_2)$, $|V(A_1 \cap A_2)| \leq 3$, $y_2 \in V(A_1 - A_2)$ and $\{a, w, t_{z_1}\} \subseteq V(A_2)$. Thus $S := \{x, z_2\} \cup V(A_1 \cap A_2)$ is a 5-cut in G separating y_2 from $B \cup A_2 \cup \{b, x_1, z_1, z\}$. Hence, by the choice of (T, S_T, A, B) (with

$|V(A)|$ minimum), $V(A_1 - A_2) = \{y_2\}$. Therefore, since G is 5-connected, $N(y_2) = S$. By the maximality of $D_b \cup D_{x_1} \cup D_{z_1}$, $R_2 - \{y_2, z\}$ has a path Q from a to w. Then $bxx_1zb \cup (ba \cup Q \cup wx_1) \cup zy_2x \cup (Y_1 \cup z_1z) \cup (Y_2 \cup z_2x) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, t, x, x_1, z.

Subcase 2.2. $y_1, y_2 \in V(A) \cup \{w\}$.

First, we show that we may assume $y_1 = w$. For, suppose $y_1, y_2 \in V(A)$. Then by Lemma 2.6.2, for each $i \in [2]$ there exists $(T_i, S_{T_i}, A_i, B_i) \in \mathcal{Q}_x$ such that $x, y_i \in V(T_i)$ and $T_i \cong K_2$ or $T_i \cong K_3$. By the assumption of this lemma, we have $T_i \cong K_3$ and $V(A) \cap S_{T_i} = \{y_i\}$. Hence, $\{b, w, x_1, z_1, z_2\} \cap V(T_i) \neq \emptyset$ for $i \in [2]$. Without loss of generality, we may assume that $y_1 \neq a$. By the symmetry between z_1 and z_2, we may also assume $z_1 \in V(T_1)$; for, otherwise, $G[T + y_1]$ or $G[T' + y_1]$ contains K_4^- and (ii) holds. Therefore, we may choose $S_{T_1} = V(T_1) \cup \{b, x_1, z_2\}$. Note the symmetry between T_1, S_{T_1} and T, S_T, and we may choose T_1, S_{T_1} as T, S_T, respectively. So we may assume $y_1 = w$ (as y_1 now plays the role of w).

Let $t \in V(B)$, and let L_1, L_2, L_3, L_4 be independent paths in $G_B = G[B + \{b, x_1, z_1, z_2\}]$ from t to z_1, z_2, b, x_1, respectively. Let $G_A := G[A + \{b, w, x_1, z_2\}]$. Note that, by the same argument as in Subcase 2.1 (with z_2 in place of z_1), we may assume that G_A is 2-connected.

We may assume that G_A does not contain independent paths from z_2, w, b to w, b, x_1, respectively; for otherwise, these paths and $T \cup bx \cup (L_1 \cup z_1x) \cup L_2 \cup L_3 \cup L_4$ form a TK_5 in G with branch vertices b, t, w, x, x_1.

Hence, since G_A is 2-connected, $wz_2 \notin E(G)$. We may assume that $wz_1 \notin E(G)$; else $G[T + z_1]$ contains K_4^- and (ii) holds. Therefore, since G is 5-connected, it follows from (2) that

$$|N(w) \cap V(A \cap D)| \geq 3.$$

Let G'_A be the graph obtained from G_A by duplicating w, b with w', b', respectively, and adding all edges from w' to $N(w)$, and from b' to $N(b)$. Then any three disjoint paths in

G'_A from $\{b, b', z_2\}$ to $\{w, w', x_1\}$ must have a path from z_2 to x_1, and we wish to apply Lemma 2.5.1.

First, we note that G'_A has no cut of size at most 2 separating $\{x_1, w, w'\}$ from $\{b, b', z_2\}$. For, otherwise, G'_A has a separation (A_1, A_2) such that $|V(A_1 \cap A_2)| \leq 2$, $\{x_1, w, w'\} \subseteq V(A_1)$ and $\{b, b', z_2\} \subseteq V(A_2)$. Note that $V(A_1 \cap A_2) \neq \{w, w'\}$ as otherwise, w would be a cut vertex in G_A. Further, $\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset$; for, otherwise, since w and w' have the same set of neighbors in G'_A, it follows from (3) that $V(A_1 \cap A_2) - \{w, w'\}$ would be a cut in G_A of size at most one. On the other hand, $V(A_1 - A_2) \subseteq \{x_1, w\}$; otherwise $(T, V(T) \cup \{z_1\} \cup V(A_1 \cap A_2), (A_1 - A_2) - w', G - (T \cup A_1)) \in \mathcal{Q}_x$ with $1 \leq |(A_1 - A_2) - w'| < |A|$, contradicting the choice of (T, S_T, A, B). However, this implies $|N(w) \cap V(A \cap D)| \leq |V(A_1 \cap A_2)| \leq 2$, a contradiction.

Hence by Lemma 2.5.1, G'_A has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (w, x_1, w'), (b, z_2, b'))$ is a ladder along some sequence $b_0 \ldots b_m$, where $b_0 = z_2$, $b_m = x_1$, and $w_0 \ldots w_n$ is the reduced sequence of $b_0 \ldots b_m$. Let P_1, P_2, P_3 be three disjoint paths in L from w, x_1, w' to b, z_2, b', respectively, and assume that they are induced in G'_A. (Let $L = G'_A$ and $J = P_2$ if (ii) of Lemma 2.5.1 holds.) Let $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$, $i \in [m]$, be the rungs in L with $a_i \in V(P_1)$ and $c_i \in V(P_3)$ for $i = 0, 1, \ldots, m$.

Since $|N(w) \cap V(A \cap D)| \geq 3$ and P_1, P_3 are induced paths in G'_A, there exists $w^* \in (N(w) \cap V(A)) - V(P_1 \cup P_3)$. We show that there exists $u \in V(P_2)$ such that $G[G_A + \{x, z_1\}]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to distinct vertices x_1, w, z_2, u_1, u_2, respectively, with $u_1, u_2 \in V(P_1 - w) \cup V(P_3 - \{b', w'\}) \cup \{x, z_1\}$, and internally disjoint from $P_1 \cup (P_3 - \{b', w'\})$. If $w^* \in V(P_2)$ then let $u = w^*$ and we see that there exist independent paths in $G_A - (V(P_1 - w) \cup V(P_3 - \{b', w'\}))$ from u to x_1, w, z_2, respectively; then the paths $Q_1, \ldots, Q_5$ exist by Lemma 2.4.11. Now suppose $w^* \notin V(P_2)$. Let $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (w, b_i, w'))$ be the rung in L containing $\{w, w', w^*\}$. Since w and w' have the same set of neighbors in G'_A, $w = a_{i-1}$ iff $w' = c_{i-1}$. If $w = a_{i-1}$ and $w' = c_{i-1}$

then $S_T^* := V(T) \cup \{b_{i-1}, b_i, z_1\}$ is a cut in G of size at most 6, and $G - S_T^*$ has a component of size smaller than $|V(A)|$, contradicting the choice of (T, S_T, A, B). So $w \neq a_{i-1}$ and $w' \neq c_{i-1}$. Suppose $R_i - x_1$ has a separation (R', R'') such that $|V(R' \cap R'')| \leq 2$, $w \in V(R' - R'')$, and $\{a_{i-1}, c_{i-1}, b_{i-1}, b_i\} - \{x_1\} \subseteq V(R'')$. Then we may assume $w' \in V(R' - R'')$ as w and w' have the same set of neighbors in G'_A. Therefore, since $|N(w) \cap V(A \cap D)| \geq 3$, $S_T^* := V(T) \cup V(R' \cap R'') \cup \{z_1\}$ is a cut in G of size at most 6, and $G - S_T^*$ has a component of size smaller than $|V(A)|$, contradicting the choice of (T, S_T, A, B). Thus we may assume, by Lemma 2.4.11, $R_i - x_1$ contains three independent paths from w to $a_{i-1}, c_{i-1}, \{b_{i-1}, b_i\} - \{x_1\}$, respectively, and internally disjoint from $\{b_{i-1}, b_i\}$. Again since w and w' have the same set of neighbors in G'_A, the parts of P_1, P_3 inside R can be modified so that the three paths in R_i correspond to $wP_1a_{i-1}, w'P_3c_{i-1}$ and a path from w to some $u \in \{b_{i-1}, b_i\} - \{x_1\}$ and internally disjoint from $P_1 \cup P_2 \cup P_3$. Thus, there exist independent paths in $G_A - (V(P_1 - w) \cup V(P_3 - \{b', w'\}))$ from u to x_1, w, z_2, respectively. Now the paths $Q_1, \ldots, Q_5$ exist by Lemma 2.4.11,.

We may assume $u_1 = z_1$ and $u_2 = x$. For, otherwise, we may assume by symmetry that $u_1 \in V(P_1)$. If $G_B - x$ has disjoint paths B_1, B_2 from z_1, b to z_2, x_1, respectively, then $T \cup bx \cup P_3 \cup B_2 \cup Q_1 \cup Q_2 \cup (Q_3 \cup B_1 \cup z_1x) \cup (Q_4 \cup u_1P_1b)$ is a TK_5 in G with branch vertices b, u, w, x, x_1. (Here we view P_3 as a path in G by identifying b', w' with b, w, respectively.) So we may assume that such B_1, B_2 do not exist. Then by Lemma 2.4.10, $(G_B - x, z_1, b, z_2, x_1)$ is planar; so the assertion of the lemma follows from Lemma 2.4.5.

We may also assume $|N(b) \cap V(B)| \leq 1$. For, suppose $|N(b) \cap V(B)| \geq 2$. Then, since G is 5-connected, $G[B + \{b, x_1, z_2\}]$ contains independent paths B_1, B_2 from b to x_1, z_2, respectively. Hence, $T \cup bx \cup P_3 \cup B_1 \cup Q_1 \cup Q_2 \cup (Q_3 \cup B_2) \cup (Q_4 \cup z_1x)$ is a TK_5 in G with branch vertices b, u, w, x, x_1, where we view P_3 as a path in G' by identifying b', w' with b, w, respectively.

Then we may assume $|N(b) \cap V(A + z_2)| \geq 3$ as otherwise, $bz_1 \in E(G)$ by (2); so $G[T' + z_1]$ contains K_4^- and (ii) holds. Let $b^* \in (N(b) \cap V(A + z_2)) - V(P_1 \cup P_3)$.

If $b^* \in V(P_2)$ let $z = b^*$ and let $P = bz$ which is internally disjoint from $P_1 \cup P_2 \cup P_3$. Now suppose $b^* \notin V(P_2)$. Let $(R_j, (b, b_{j-1}, b'), (a_j, b_j, c_j))$ be the rung in L containing $\{b, b,' b^*\}$. Since b and b' have the same set of neighbors in G'_A, $b = a_j$ iff $b' = c_j$. If $b = a_j$ and $b' = c_j$ then, since $az_1 \notin E(G)$, $S_T^* := V(T') \cup \{b_{j-1}, b_j, z_1\}$ is a cut in G of size 6 and $G - S_T^*$ has a component of size smaller than $|V(A)|$, contradicting the choice of (T, S_T, A, B). So $b \neq a_j$ and $b' \neq c_j$. We claim that $P_1 \cap R_j$ and $P_3 \cap R_j$ may be modified so that G_A contains a path P from b to some $z \in V(P_2)$ and internally disjoint from $P_1 \cup P_2 \cup (P_3 - \{b', w'\})$. If R_j contains three independent paths from b to $a_j, c_j, \{b_{j-1}, b_j\}$, respectively, and internally disjoint from $\{a_j, c_j, b_{j-1}, b_j\}$, then $P_1 \cap R_j, P_3 \cap R_j$ can be modified so that the three paths in R_j correspond to $bP_1a_j, b'P_3c_j$ and a path P from b to $z \in \{b_{j-1}, b_j\}$ and internally disjoint from $P_1 \cup P_2 \cup (P_3 - \{b', w'\})$. So assume that such three paths in R_j do not exist. Then by the existence of bP_1a_j and $b'P_3c_j$ and by Lemma 2.4.11, R_j has no three independent paths from b to $\{a_j, c_j, b_{j-1}, b_j\}$ and internally disjoint from $\{a_j, c_j, b_{j-1}, b_j\}$. Thus R_j has a separation (A_1, A_2) with $|V(A_1 \cap A_2)| \leq 2$, $V(A_1 \cap A_2) \subseteq V(P_1 \cup P_3)$, $b, b^* \in V(A_1 - A_2)$ and $\{a_j, c_j, b_{j-1}, b_j\} \subseteq V(A_2)$. Since b' is a copy of b, we may assume $b' \in V(A_1 - A_2)$. Now, since $az_1 \notin E(G)$, $V(A_1 \cap A_2) \cup \{x, b, z_1\}$ is a cut in G; so $V(A_1) = V(A_1 \cap A_2) \cup \{b, b', b^*\}$ by the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Then $b^*x, b^*z_1 \in E(G)$ (as G is 5-connected); so $G[\{x, b^*, b, z_1\}]$ contains K_4^-, and (ii) holds.

Suppose $R_i \neq R_j$. Since G is 5-connected, $G[B + \{b, x_1\}]$ has a path B_1 from b to x_1. Since Q_3 is internally disjoint from $P_1 \cup P_3$, we may assume that $z \in V(Q_3)$ and P is also internally disjoint from Q_3. Hence, $T \cup bx \cup P_3 \cup B_1 \cup Q_1 \cup Q_2 \cup (uQ_3z \cup P) \cup (Q_4 \cup z_1x)$ is a TK_5 in G' with branch vertices b, u, w, x, x_1, where we view P_3 as a path in G by identifying b', w' with b, w, respectively.

So $R_i = R_j$. Then $a_{i-1} = b$ and $c_{i-1} = b'$. Recall $bw \notin E(G)$ (by (2)). Since w and w' (respectively, b and b') have the same set of neighbors in G'_A, it follows from Lemma 2.5.3 that $b_{i-1} = b_i$. Then $\{b, b_i, w, x, z_1\}$ is a cut in G separating $P_1 \cup (P_3 - \{b', w'\})$ from

$B \cup J$. Since $bw \notin E(G)$, $|V(P_1 \cup (P_3 - \{b', w'\}))| \geq 2$. This contradicts the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. ∎

2.7 Interactions between quadruples

In this section, we explore the structure of G by considering a quadruple (T, S_T, A, B) with $|V(A)|$ minimum and a quadruple $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. The lemma below allows us to assume that if $T \cap C = \emptyset$ then $A \cap C = \emptyset$.

Lemma 2.7.1 *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $T \cap C = \emptyset$. Then $A \cap C = \emptyset$, or one of the following holds:*

(i) *G contains a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(iii) *There exist $x_1, x_2, x_3 \in N(x)$ such that for any $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*

Proof. We may assume $T \cong K_3$ (by Lemma 2.6.3) and $T' \cong K_3$ (by Lemma 2.6.4). Suppose $A \cap C \neq \emptyset$.

Then $|(S_T \cup S_{T'}) - V(B \cup D)| \geq 7$; otherwise $(T', (S_{T'} \cup S_T) - V(B \cup D), A \cap C, B \cup D) \in \mathcal{Q}_x$ and $1 \leq |V(A \cap C)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence $|(S_T \cup S_{T'}) - V(A \cup C)| = 5$, as $|S_T| = |S_{T'}| = 6$. Since $T \cap C = \emptyset$, $V(T) \subseteq (S_T \cup S_{T'}) - V(A \cup C)$.

Suppose $|V(B \cap D)| \geq 2$. Then G has a separation (G_1, G_2) such that $VG_1 \cap G_2) = (S_T \cup S_{T'}) - V(A \cup C)$ and $|V(G_i)| \geq 7$. So the assertion of this lemma follows from Lemma 2.4.6.

Hence, we may assume $|V(B \cap D)| \leq 1$. Therefore, by the minimality of $|V(A)|$, $|S_T \cap V(D)| \geq |S_{T'} \cap V(A)|$. But this implies that $|S_T| \geq |(S_T \cup S_{T'}) - V(B \cup D)| \geq 7$, a contradiction. ∎

We need a lemma for finding paths to deal with a special case when $A \cap C = \emptyset$ for quadruples $(T, S_T, A, B), (T', S_{T'}, C, D) \in \mathcal{Q}_x$.

Lemma 2.7.2 *Let G be a 5-connected nonplanar graph and $x \in V(G)$, and suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Let $V(T) = \{x, x_1, x_2\}$ and $V(T') = \{x, a, b\}$ with $a \in V(A)$. Suppose $A \cap C = \emptyset$, $|S_T| = 6 = |S_{T'}|$, $V(T) \subseteq S_T - V(C)$, $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$, and $(S_T \cup S_{T'}) - V(B \cup C \cup T \cup T') = \{x_3, x_4\}$. Then G contains K_4^-, or the following statements hold:*

- (i) *$N(b) \cap V(A-a) \neq \emptyset$ and if $t \in N(b) \cap V(A-a)$ then $G[(A-a)+\{b, x_1, x_2, x_3, x_4\}]$ has independent paths from t to b, x_1, x_2, x_3, x_4, respectively, and*
- (ii) *if $b \in S_T$ then $G[A+\{b, x_1, x_2\}]$ has independent paths from b to x_1, x_2, respectively.*

Proof. First, we note that $N(b) \cap V(A - a) \neq \emptyset$. For, otherwise, $(T, (S_T \cup S_{T'}) - V(B \cup C) - \{b\}, A - a, G[B \cup C + b]) \in \mathcal{Q}_x$. By the choice of (T, S_T, A, B) that $|V(A)|$ is minimum, we must have $V(A - a) = \emptyset$. So G contains K_4^- by Lemma 2.6.1.

To complete the proof of (i), let $t \in N(b) \cap V(A - a)$. If $G[(A - a) + \{x_1, x_2, x_3, x_4\}]$ has four independent paths from t to x_1, x_2, x_3, x_4, respectively, then these four paths and tb give the desired five paths. So we may assume that such four paths do not exist. Then $G[(A - a) + \{x_1, x_2, x_3, x_4\}]$ has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $t \in V(G_1 - G_2)$ and $\{x_1, x_2, x_3, x_4\} \subseteq V(G_2)$. Hence, $(T', V(T') \cup V(G_1 \cap G_2), G_1 - G_2, G - T' - G_1) \in \mathcal{Q}_x$ and $1 \leq |V(G_1 - G_2)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of (T, S_T, A, B).

To prove (ii), let $b \in S_T$ and assume that the two paths in (ii) do not exist. Note that if $b \in V(T)$ then $T \cup T'$ contains K_4^-. So we may assume $b \notin V(T)$. Then, $G[A+\{b, x_1, x_2\}]$

has a separation (G_1, G_2) such that $|V(G_1) \cap V(G_2)| \leq 1$, $b \in V(G_1) - V(G_2)$ and $\{x_1, x_2\} \subseteq V(G_2)$. Since $N(b) \cap V(A - a) \neq \emptyset$ and $|V(G_1) \cap V(G_2)| \leq 1$, $|V(G_1 - G_2)| \geq 2$. Let $S_{bx} = (S_T - \{x_1, x_2\}) \cup V(G_1 \cap G_2)$, and let $F = G_1 - S_{bx}$. Then $|V(F)| \geq 1$ as $|V(G_1 - G_2)| \geq 2$. If $|V(F)| \geq 2$ then $(bx, S_{bx}, F, G - S_{bx} - F) \in \mathcal{Q}_x$ with $2 \leq |V(F)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. So assume $|V(F)| = 1$ and let $v \in V(F)$. Since G is 5-connected, v is adjacent to all vertices in S_{bx}. If $v \neq a$ then $V(G_1 \cap G_2) = \{a\}$; so $G[\{a, b, v, x\}]$ contains K_4^-. Now assume $v = a$. Let $w \in V(G_1 \cap G_2)$. Since $N(b) \cap V(A - a) \neq \emptyset$, $bw \in E(G)$. So $G[\{a, b, w, x\}]$ contains K_4^-. ∎

In the next two lemmas, we consider the case when quadruples (T, S_T, A, B) and $(T', S_{T'}, C, D)$ may be chosen so that $|V(T' \cap A)| = 2$.

Lemma 2.7.3 *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum. Suppose there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $T' \cong K_3$ and $|V(T' \cap A)| = 2$. Then one of the following holds:*

(i) *G contains a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(iii) *There exist $x_1, x_2, x_3 \in N(x)$ such that for any $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*

(iv) *$|S_T \cap S_{T'}| = 1$, $|S_{T'} \cap V(B)| = 2$, and either $|S_T \cap V(C)| = 2$ and $T \cap C = \emptyset$ or $|S_T \cap V(D)| = 2$ and $T \cap D = \emptyset$.*

Proof. We may assume $T \cong K_3$ (by Lemma 2.6.3). We may also assume that $|S_T| = |S_{T'}| = 6$; for, otherwise, (i) or (ii) or (iii) follows from Lemma 2.4.6. We may further assume $|V(A)| \geq 5$; as otherwise, by Lemma 2.6.1, G contains K_4^- and (ii) holds.

Let $T' = \{a, b, x\}$ with $a, b \in V(A)$. By symmetry, assume $T \cap C = \emptyset$. Then, by Lemma 2.7.1, we may assume $A \cap C = \emptyset$. Now $B \cap C \neq \emptyset$; for, otherwise, $|V(C)| = |S_T \cap V(C)| \leq 3$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, $S_T \cap V(C) \neq \emptyset$ as $S_{T'} - \{a, b\}$ is not a cut in G. Moreover, $A \cap D \neq \emptyset$; for otherwise, $|V(A) \cap S_{T'}| = 5$ and, hence, $|S_{T'} \cap S_T| = 1$ and $|S_{T'} \cap V(B)| = 0$; so $(S_T \cup S_{T'}) - V(A \cup D)$ is a cut in G of size at most 4 and separating $B \cap C$ from $A \cup D$, a contradiction.

We claim that $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. First, note that $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$; otherwise, $(T', (S_T \cup S_{T'}) - V(B \cup C), A \cap D, B \cup C) \in \mathcal{Q}_x$ and $1 \leq |V(A \cap D)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)$ is minimum. Also note that $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$ since $B \cap C \neq \emptyset$ and G is 5-connected. Thus the claim follows from the fact that $|(S_T \cup S_{T'}) - V(B \cup C)| + |(S_T \cup S_{T'}) - V(A \cup D)| = |S_T| + |S_{T'}| = 12$.

We may assume that $|S_T \cap V(C)| \neq 1$ or $|S_{T'} \cap V(A)| \neq 2$. For, suppose $S_T \cap V(C) = \{c\}$ and $S_{T'} \cap V(A) = \{a, b\}$. If $a, b \in N(c)$ then $G[T' + c]$ contains K_4^- and (ii) holds. So by the symmetry between a and b, we may assume that $ca \notin E(G)$. Then $(T, (S_T - c) \cup \{b\}, A - b, G[B + c]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

We may also assume $T \cap D \neq \emptyset$; for, otherwise, since $A \cap D \neq \emptyset$, (i) or (ii) or (iii) follows from Lemma 2.7.1. Therefore, $S_T \cap V(D) \neq \emptyset$. Note that $1 \leq |S_T \cap S_{T'}| \leq 4$, and we distinguish four cases according to $|S_T \cap S_{T'}|$.

Suppose $|S_T \cap S_{T'}| = 4$. Then $S_{T'} \cap V(B) = \emptyset$ and $|S_T \cap V(C)| = |S_T \cap V(D)| = 1$. Therefore, by the minimality of $|V(A)|$, $B \cap D \neq \emptyset$. Hence, $S_T - V(C)$ is a 5-cut in G and $V(T) \subseteq S_T - V(C)$. By the choice of (T, S_T, A, B) that $|V(A)|$ is minimum, $|V(B \cap D)| \geq 5$. Now (i) or (ii) or (iii) follows from Lemma 2.4.6.

Consider $|S_T \cap S_{T'}| = 3$. Suppose for the moment $S_{T'} \cap V(B) = \emptyset$. Then $|S_T \cap V(C)| = 2$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. So $B \cap D = \emptyset$ as otherwise $S_T - V(C)$ would be a 4-cut in G. However, this implies $|V(D)| < |V(A)|$, contradicting the choice of

(T, S_T, A, B) that $|V(A)|$ is minimum. So $S_{T'} \cap V(B) \neq \emptyset$. Therefore, since $|S_{T'}| = 6$, we have $|S_{T'} \cap V(B)| = 1$ and $S_{T'} \cap V(A) = \{a, b\}$. Since $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$, $|S_T \cap V(C)| = 1$. This is a contradiction, as we have $|S_T \cap V(C)| \neq 1$ or $|S_{T'} \cap V(A)| \neq 2$.

Now let $|S_T \cap S_{T'}| = 2$. First, assume $|S_T \cap V(C)| = 1$. Then $|S_{T'} \cap V(B)| = 2$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$) and, hence, $|S_{T'} \cap V(A)| = 2$ (as $|S_{T'}| = 6$), a contradiction. So we may assume that $|S_T \cap V(C)| \geq 2$, which implies $|S_{T'} \cap V(B)| \leq 1$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. Hence, since $|S_T| = |S_{T'}| = 6$, $|S_{T'} \cap V(A)| \geq 3$ and $|S_T \cap V(D)| \leq 2$. Therefore, by the minimality of $|V(A)|$, $B \cap D \neq \emptyset$. Thus $(S_T \cap S_{T'}) - V(A \cup C)$ is a 5-cut in G and contains $V(T)$. So $|V(B \cap D)| \geq 5$ by the minimality of $|V(A)$. Now (i) or (ii) or (iii) follows from Lemma 2.4.6.

Finally, assume $|S_T \cap S_{T'}| = 1$. If $|S_{T'} \cap V(B)| = 2$ then $|S_T \cap V(C)| = 2$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$); so (iv) holds. If $|S_{T'} \cap V(B)| = 3$ then $|S_T \cap V(C)| = 1$ (since $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$) and $S_{T'} \cap V(A) = \{a, b\}$ (as $|S_{T'}| = 6$), a contradiction. Hence, we may assume $|S_{T'} \cap V(B)| \leq 1$. Then $|S_T \cap V(C)| \geq 3$ (since $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$), $|S_{T'} \cap V(A)| \geq 4$, and $|(S_T \cup S_{T'}) - V(A \cup C)| \leq 4$. Hence, since G is 5-connected, $B \cap D = \emptyset$; so $|V(D)| < |V(A)|$. However, this shows that $(T', S_{T'}, D, C)$ contradicts the choice of (T, S_T, A, B). ∎

Next, we take care of the case when (iv) of Lemma 2.7.3 holds.

Lemma 2.7.4 *Let G be a 5-connected nonplanar graph and $x \in V(G)$, and suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $T \cap C = \emptyset$, $S_T \cap S_{T'} = \{x\}$ and $|S_T \cap V(C)| = |S_{T'} \cap V(B)| = 2$. Then one of the following holds:*

(i) G contains a TK_5 in which x is not a branch vertex.

(ii) G contains K_4^-.

(iii) There exist $x_1, x_2, x_3 \in N(x)$ *such that, for any* $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ *contains* TK_5.

Proof. We may assume $T \cong K_3$ (by Lemma 2.6.3) and $T' \cong K_3$ (by Lemma 2.6.4). By Lemma 2.6.1, we may assume $|V(A)| \geq 5$. We may further assume that $|S_T| = |S_{T'}| = 6$; for, otherwise, the assertion follows from Lemma 2.4.6.

Let $V(T) = \{x, x_1, x_2\}$, $V(T') = \{x, a, b\}$, $S_T \cap V(C) = \{p_1, p_2\}$, $S_{T'} \cap V(B) = \{c_1, c_2\}$, $S_{T'} \cap V(A) = \{a, b, q\}$, and $S_T \cap V(D) = \{x_1, x_2, w\}$. Since $T \cap C = \emptyset$, we may assume by Lemma 2.7.1 that $A \cap C = \emptyset$. Then $B \cap C \neq \emptyset$ by the minimality of $|V(A)|$.

We may assume $N(p_1) \cap V(A) = \{a, q\}$ and $N(p_2) \cap V(A) = \{b, q\}$. To see this, for $i \in [2]$, let $S_i := (S_T - \{p_i\}) \cup (N(p_i) \cap \{a, b, q\})$ which is a cut in G and containing $V(T)$. If $N(p_i) \cap \{a, b, q\} = \emptyset$ then $|S_i| = 5$ and the assertion of this lemma follows from Lemma 2.4.6. If $|N(p_i) \cap \{a, b, q\}| = 1$ then $(T, S_i, A - (N(p_i) \cap \{a, b, q\}), S_i, G[B + p_i]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, we may assume that $|N(p_i) \cap \{a, b, q\}| \geq 2$ for $i \in [2]$. We may assume $\{a, b\} \not\subseteq N(p_i)$ for $i \in [2]$; as otherwise, $G[T' + p_i]$ contains K_4^- and (ii) holds. Moreover, $N(p_1) \cap \{a, b, q\} \neq N(p_2) \cap \{a, b, q\}$, as otherwise, $S := (S_T - \{p_1, p_2\}) \cup (N(p_1) \cap \{a, b, q\})$ is a cut in G containing $V(T)$; so $(T, S, A - (N(p_1) \cap \{a, b, q\}), G[B + \{p_1, p_2\}]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) with $|V(A)|$ minimum. Hence, we may assume $N(p_1) \cap V(A) = \{a, q\}$ and $N(p_2) \cap V(A) = \{b, q\}$.

Note that $N(x_i) \cap V(B) \neq \emptyset$ for $i \in [2]$; for, otherwise, $S := V(T') \cup \{q, x_{3-i}, w\}$ is a cut in G, and $(T', S, G[(A \cap D) + x_i], G[B + \{p_1, p_2\}]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Moreover, we may assume $N(w) \cap V(B) \neq \emptyset$; as otherwise, $S_T - \{w\}$ is a 5-cut in G and $V(T) \subseteq S_T - \{w\}$, and the assertion of this lemma follows from Lemma 2.4.6.

We wish to prove (iii) with $x_3 = b$. Let $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$ be distinct. Choose $v \in \{y_1, y_2\} - \{a\}$. We may assume $v \notin \{p_1, p_2\}$, as otherwise $G[T' + v]$ contains K_4^- and (ii) holds. By Lemma 2.7.2, we may choose $t \in N(b) \cap V(A - a)$

such that $G[(A-a)+\{b,q,x_1,x_2,w\}]$ has independent paths P_1,P_2,P_3,P_4,P_5 from t to b,x_1,x_2,w,q respectively. We distinguish four cases according to the location of v.

Case 1. $v \in V(B)$.

Let W be the component of B containing v. First, suppose $N(x_i)\cap W \neq \emptyset$ for $i\in[2]$. Then there exists $v^* \in V(W)$ such that $G[W+\{x_1,x_2\}]$ has three independent paths from v^* to v,x_1,x_2, respectively. Hence by Lemma 2.4.11, $G[W+(S_T-\{x\})]$ has independent paths Q_1,Q_2,Q_3,Q_4 from v^* to v,x_1,x_2,u, respectively, and internally disjoint from S_T, where $u\in S_T-\{x,x_1,x_2\}$. If $u=w$ then $T\cup(P_1\cup bx)\cup P_2\cup P_3\cup(Q_1\cup vx)\cup Q_2\cup Q_3\cup(Q_4\cup P_4)$ is a TK_5 in G' with branch vertices t,v^*,x,x_1,x_2. If $u=p_i$ for some $i\in[2]$ then $T\cup(P_1\cup bx)\cup P_2\cup P_3\cup(Q_1\cup vx)\cup Q_2\cup Q_3\cup(Q_4\cup p_iq\cup P_5)$ is a TK_5 in G' with branch vertices t,v^*,x,x_1,x_2.

Thus, we may assume that $N(x_1)\cap W=\emptyset$. Since G is 5-connected, $N(x_2)\cap W\neq\emptyset$. So $G[W+(S_T-\{x_1\})]$ has independent paths Q_1,Q_2,Q_3,Q_4,Q_5 from v to x,x_2,w,p_1,p_2, respectively. Clearly, we may assume that $Q_1=vx$. Since $N(x_1)\cap V(B)\neq\emptyset$, let W' be a component of B with $N(x_1)\cap V(W')\neq\emptyset$. Since G is 5-connected, there exists $i\in[2]$ such that $N(p_i)\cap V(W')\neq\emptyset$. Hence, $G[W'+\{x_1,p_i\}]$ has a path R from x_1 to p_i, and, by symmetry, assume R is from x_1 to p_1. Now $T\cup(P_1\cup bx)\cup P_2\cup P_3\cup Q_1\cup Q_2\cup(Q_3\cup P_4)\cup(Q_4\cup R)$ is a TK_5 in G' with branch vertices t,v,x,x_1,x_2.

Case 2. $v\in V(A\cap D)$.

First, we show that $G[(A\cap D)+\{q,w,x,x_1,x_2\}]$ has independent paths P_1',P_2',P_3',P_4',P_5' from v to q,x,x_1,x_2,w, respectively (and we may assume that $P_2'=vx$). This is clear if $G[(A\cap D)+\{q,w,x_1,x_2\}]$ has independent paths from v to q,x_1,x_2,w, respectively. So we may assume that $G[(A\cap D)+\{q,w,x_1,x_2\}]$ has a separation (G_1,G_2) such that $|V(G_1\cap G_2)|\leq 3$, $v\in V(G_1-G_2)$ and $\{q,w,x_1,x_2\}\subseteq V(G_2)$. Then $S:=V(T')\cup V(G_1\cap G_2)$ is a cut in G, and $(T',S,G_1-G_2,G-S-G_1)\in\mathcal{Q}_x$, contradicting the choice of (T,S_T,A,B) that $|V(A)|$ is minimum.

Suppose B has a component W such that $N(x_i) \cap W \neq \emptyset$ for $i \in [2]$. Then there exists $z \in V(W)$ such that $G[W + \{x_1, x_2\}]$ has independent paths from z to x_1, x_2, respectively. Hence by Lemma 2.4.11, $G[W + (S_T - \{x\})]$ has four independent paths Q_1, Q_2, Q_3, Q_4 from z to x_1, x_2, u_1, u_2, respectively, and internally disjoint from S_T, where $u_1, u_2 \in \{w, p_1, p_2\}$ are distinct. If $\{u_1, u_2\} = \{w, p_1\}$ then we may assume $u_1 = w$ and $u_2 = p_1$; now $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_5') \cup (Q_4 \cup p_1abx)$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z. If $\{u_1, u_2\} = \{w, p_2\}$ then we may assume $u_1 = w$ and $u_2 = p_2$; now $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_5') \cup (Q_4 \cup p_2bx)$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z. So assume $\{u_1, u_2\} = \{p_1, p_2\}$. We may further assume $u_i = p_i$ for $i \in [2]$. Then $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup p_1q \cup P_1') \cup (Q_4 \cup p_2bx)$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z.

Hence, we may assume that no component of B contains neighbors of both x_1 and x_2. Since G is 5-connected, we may assume by symmetry that Z is a component of B such that $N(x_1) \cap V(Z) = \emptyset$ and $N(x_2) \cap V(Z) \neq \emptyset$. Again, since G is 5-connected, $G[Z + (S_T - \{x_1\})]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $z \in V(Z)$ to x_2, w, p_1, p_2, x, respectively. Since $N(x_1) \cap V(B) \neq \emptyset$, let Z' be a component of B with $N(x_1) \cap Z' \neq \emptyset$. Then $N(x_2) \cap V(Z') = \emptyset$. So $G[Z' + \{x_1, p_1\}]$ contains a path R from x_1 to p_1. Now $T \cup P_2' \cup P_3' \cup P_4' \cup (Q_4 \cup p_2bx) \cup Q_1 \cup (Q_3 \cup R) \cup (Q_2 \cup P_5')$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z.

Case 3. $v = q$.

Suppose B has a component Z such that $\{w, x_1, x_2\} \subseteq N(Z)$. Then there exists $z \in V(Z)$ such that $G[Z + \{w, x_1, x_2\}]$ has independent paths from z to w, x_1, x_2, respectively. By Lemma 2.4.11, $G[Z + (S_T - \{x\})]$ has independent paths Q_1, Q_2, Q_3, Q_4 from z to x_1, x_2, w, u, respectively, and internally disjoint from S_T, where $u \in \{p_1, p_2\}$. Let $S = Q_4 \cup p_1abx$ if $u = p_1$ and $S = Q_4 \cup p_2bx$ if $u = p_2$. Then $T \cup Q_1 \cup Q_2 \cup S \cup (P_4 \cup Q_3) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z.

So we may assume that no component of B is adjacent to all of x_1, x_2 and w. Since $N(w) \cap V(B) \neq \emptyset$, there exists a component Z of B such that $N(w) \cap V(Z) \neq \emptyset$. Since G is 5-connected, we may assume by symmetry that $N(x_2) \cap V(Z) \neq \emptyset$. Then $N(x_1) \cap V(Z) = \emptyset$. Since G is 5-connected, $G[Z + (S_T - \{x_1\})]$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $z \in V(Z)$ to x_2, w, p_1, p_2, x, respectively. Since $N(x_1) \cap V(B) \neq \emptyset$, there exists some component Z' of B with $N(x_1) \cap V(Z') \neq \emptyset$. Hence, $N(x_2) \cap V(Z') = \emptyset$ or $N(w) \cap V(Z') = \emptyset$; so $G[Z' + \{x_1, p_1\}]$ contains a path R from x_1 to p_1. Now $T \cup Q_1 \cup (Q_3 \cup R) \cup (Q_4 \cup p_2bx) \cup (P_4 \cup Q_2) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z.

Case 4. $v = w$.

Suppose B has a component Z such that $\{w, x_1, x_2\} \subseteq N(Z)$. Then there exists $z \in V(Z)$ such that $G[Z + \{w, x_1, x_2\}]$ has three independent paths from z to w, x_1, x_2, respectively. Hence, by Lemma 2.4.11, $G[Z + (S_T - \{x\})]$ has independent paths Q_1, Q_2, Q_3, Q_4 from z to x_1, x_2, w, u, respectively, and internally disjoint from S_T, where $u = p_i$ for some $i \in [2]$. Then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup wx) \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (P_5 \cup qp_i \cup Q_4)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z.

Hence, we may assume that no component of B is adjacent to all of w, x_1, x_2. Since $N(w) \cap V(B) \neq \emptyset$, B has a component Z such that $N(w) \cap V(Z) \neq \emptyset$. Since G is 5-connected, we may assume by symmetry that $N(x_2) \cap V(Z) \neq \emptyset$. Then $N(x_1) \cap V(Z) = \emptyset$. Since G is 5-connected, $G[Z + (S_T - \{x_1\})]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from z to x_2, w, p_1, p_2, x, respectively. Since $N(x_1) \cap V(B) \neq \emptyset$, B has a component Z' such that $N(x_1) \cap V(Z') \neq \emptyset$. Then $N(x_2) \cap V(Z') = \emptyset$ or $N(w) \cap V(Z') = \emptyset$; so $G[Z' + \{x_1, p_1\}]$ contains a path R from x_1 to p_1. Now $T \cup Q_1 \cup (Q_2 \cup wx) \cup (Q_3 \cup R) \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (P_5 \cup qp_2 \cup Q_4)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z. ∎

We end this section with the following lemma which deals with another special case when $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$, and $A \cap C = \emptyset$.

Lemma 2.7.5 *Let G be a 5-connected nonplanar graph and $x \in V(G)$ such that for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $A \cap C = \emptyset$, $|S_T| = 6$, $|S_{T'}| = 6$, $V(T') \cap S_T = \{x, b\}$, $V(T' \cap A) = S_{T'} \cap V(A) = \{a\}$ and $V(C) \cap S_T = \emptyset$. Then, one of the following holds:*

(i) *G contains a TK_5 in which x is not a branch vertex.*

(ii) *G contains K_4^-.*

(iii) *There exist distinct $x_1, x_2 \in N(x)$ such that for any distinct $y_1, y_2 \in N(x) - \{b, x_1, x_2\}$, $G' := G - \{xv : v \notin \{x_1, x_2, b, y_1, y_2\}\}$ contains TK_5.*

Proof. By assumption, $V(T') = \{a, b, x\}$ with $a \in V(A)$ and $b, x \in S_T \cap S_{T'}$. Let $V(T) = \{x, x_1, x_2\}$ and $S_T = \{b, x, x_1, x_2, x_3, x_4\}$. We wish to prove (iii) with $x_3 = b$; so let $y_1, y_2 \in N(x) - \{b, x_1, x_2\}$ be distinct. Let $v \in \{y_1, y_2\} - \{a\}$.

Note that $B \cap C \neq \emptyset$ as $S_{T'}$ is a cut. So $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$. Moreover, we may assume $A \cap D \neq \emptyset$ by Lemma 2.6.1. So $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$ by the minimality of $|V(A)|$. Since $|S_T| = |S_{T'}| = 6$,

$$|(S_T \cup S_{T'}) - V(A \cup D)| = 5 \text{ and } |(S_T \cup S_{T'}) - V(B \cup C)| = 7.$$

We may assume that $N(x_i) \cap V(B) \neq \emptyset$ for $i \in [2]$. For, suppose this is not true and by symmetry assume $N(x_1) \cap V(B) = \emptyset$. Let $S = (S_T - \{x_1\}) \cup \{a\}$, $C' = B$, and $D' = G[(A - a) + x_1]$. Then $(T', S, C', D') \in \mathcal{Q}_x$. We now apply Lemma 2.6.6 to (T, S_T, A, B) and (T', S, C', D'). Note that $|S \cap S_T| = 5$, $V(A \cap C') = S_T \cap V(C') = S \cap V(B) = V(B \cap D') = \emptyset$, and $|S \cap V(A)| = |S_T \cap V(D')| = |V(T \cap D')| = 1$. To verify the other condition in Lemma 2.6.6, let $(H, S_H, C_H, D_H) \in \mathcal{Q}_x$ such that $H \cong K_2$ or $H \cong K_3$. Then we may assume that $H \cong K_3$ when $H \cap A \neq \emptyset$ (by Lemma 2.6.4) and that $|V(H \cap A)| \leq 1$ (by Lemmas 2.7.3 and 2.7.4). Therefore, the assertion of this lemma follows from Lemma 2.6.6. Hence, we may assume $N(x_i) \cap B \neq \emptyset$ for $i \in [2]$.

We may assume that for any component W of B, $N(b) \cap W \neq \emptyset$; for, otherwise, $S_T - \{b\}$ is a 5-cut in G, and the assertion of this lemmas follows from Lemma 2.4.6. We consider three cases according to the location of v.

Case 1. $v \in V(B)$.

Let B_v be the component of B containing v. First, suppose $N(x_i) \cap V(B_v) \neq \emptyset$ for $i \in [2]$. Then $G[B_v + \{x_1, x_2\}]$ has independent paths from some $v^* \in V(B_v)$ to v, x_1, x_2, respectively. Thus, by Lemma 2.4.11, $G[B_v + S_T - x]$ has independent paths P_1, P_2, P_3, P_4 from v^* to v, x_1, x_2, u, respectively, and internally disjoint from S_T, where $u \in \{b, x_3, x_4\}$. Suppose $u = b$. By Lemma 2.7.2, we may assume that $G[A + \{b, x_1, x_2\}]$ contains independent paths R_1, R_2 from b to x_1, x_2, respectively. Then $T \cup R_1 \cup R_2 \cup bx \cup (P_1 \cup vx) \cup P_2 \cup P_3 \cup P_4$ is a TK_5 in G' with branch vertices b, v^*, x, x_1, x_2. So we may assume by symmetry that $u = x_3$. By Lemma 2.7.2 again, we may choose $t \in N(b) \cap V(A-a)$ and let Q_1, Q_2, Q_3, Q_4, Q_5 be independent paths in $G[(A-a)+\{b, x_1, x_2, x_3, x_4\}]$ from t to b, x_1, x_2, x_3, x_4, respectively. Then, $T \cup (Q_1 \cup bx) \cup Q_2 \cup Q_3 \cup (P_1 \cup vx) \cup P_2 \cup P_3 \cup (P_4 \cup Q_4)$ is a TK_5 in G' with branch vertices t, v^*, x, x_1, x_2.

Therefore, we may assume by symmetry that $N(x_1) \cap V(B_v) = \emptyset$. Since G is 5-connected, $G[B_v + S_T - x_1]$ has independent paths P_1, P_2, P_3, P_4, P_5 from v to x, b, x_2, x_3, x_4, respectively, and we may assume that $P_1 = vx$. Since $N(x_1) \cap V(B) \neq \emptyset$, B has a component B_{x_1} such that $N(x_1) \cap V(B_{x_1}) \neq \emptyset$. Again, since G is 5-connected, $N(x_j) \cap V(B_{x_1}) \neq \emptyset$ for some $j \in \{3, 4\}$, and we may assume $j = 3$. Then $G[B_{x_1} + \{x_1, x_3\}]$ contains a path Q from x_1 to x_3. Let $t \in N(b) \cap V(A - a)$. By Lemma 2.7.2, we may assume that $G[(A - a) + \{b, x_1, x_2, x_3, x_4\}]$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from t to b, x_1, x_2, x_3, x_4, respectively. Then $T \cup (Q_1 \cup bx) \cup Q_2 \cup Q_3 \cup (P_5 \cup Q_5) \cup (P_4 \cup Q) \cup P_1 \cup P_3$ is a TK_5 in G' with branch vertices t, v, x, x_1, x_2.

Case 2. $v \in V(A \cap D)$.

We claim that $G[(A-a)+\{x,x_1,x_2,x_3,x_4\}]$ has independent paths P_1,P_2,P_3,P_4,P_5 from v to x,x_1,x_2,x_3,x_4, respectively (and we may assume $P_1=vx$). This is clear if $G[(A-a)+\{x_1,x_2,x_3,x_4\}]$ has independent paths from v to x_1,x_2,x_3,x_4, respectively; so we may assume such paths do not exist. Then there exists a separation (G_1,G_2) in $G[(A-a)+\{x_1,x_2,x_3,x_4\}]$ such that $|V(G_1\cap G_2)|\leq 3$, $v\in V(G_1-G_2)$, and $\{x_1,x_2,x_3,x_4\}\subseteq V(G_2)$. Let $S:=V(G_1\cap G_2)\cup V(T')$, which is a cut in G of size at most 6. Since G is 5-connected, $|V(G_1\cap G_2)|\geq 2$. Then, $(T',S,G_1-G_2,(G-S)-G_1)\in\mathcal{Q}_x$ and $1\leq|V(G_1-G_2)|\leq|V(A-a)|<|V(A)|$, contradicting the choice of (T,S_T,A,B) that $|V(A)|$ is minimum.

Suppose that B has a component W such that $N(x_i)\cap V(W)\neq\emptyset$ for $i\in[2]$. Then there exists $w\in V(W)$ such that $G[W+b]$ has independent paths from w to x_1,x_2,b, respectively. By Lemma 2.4.11, $G[B+S_T]$ has independent paths Q_1,Q_2,Q_3,Q_4, Q_5 from w to x_1,x_2,b,u_1,u_2, respectively, and internally disjoint from S_T, where $u_1,u_2\in\{x,x_3,x_4\}$ are distinct. By symmetry, we may assume $u_1=x_3$. Then $T\cup P_1\cup P_2\cup P_3\cup Q_1\cup Q_2\cup(Q_3\cup bx)\cup(Q_4\cup P_4)$ is a TK_5 in G' with branch vertices v,w,x,x_1,x_2.

Hence, we may assume that no component of B is adjacent to both x_1 and x_2. Let W be a component of B such that $N(x_2)\cap V(W)\neq\emptyset$. Then $N(x_1)\cap V(W)=\emptyset$. Since G is 5-connected, $G[W+S_T-x_1]$ has independent paths Q_1,Q_2,Q_3,Q_4,Q_5 from some $w\in V(W)$ to b,x_2,x_3,x_4,x, respectively. Since $N(x_1)\cap V(B)\neq\emptyset$, B has a component B_x such that $N(x_1)\cap V(B_x)\neq\emptyset$. Then $N(x_2)\cap V(B_x)=\emptyset$. Again, since G is 5-connected, $G[B_x+\{x_1,x_3\}]$ contains a path R from x_1 to x_3. Now $T\cup P_1\cup P_2\cup P_3\cup(Q_1\cup bx)\cup Q_2\cup(Q_3\cup R)\cup(Q_4\cup P_5)$ is a TK_5 in G' with branch vertices v,w,x,x_1,x_2.

Case 3. $v\in S_T$.

We may assume that $v=x_3$. By Lemma 2.7.2, we may assume $t\in N(b)\cap V(A-a)$ and $G[(A-a)+\{b,x_1,x_2,x_3,x_4\}]$ has independent paths P_1,P_2,P_3,P_4,P_5 from t to b,x_1,x_2, x_3,x_4, respectively, with $P_1=tb$. Also by Lemma 2.7.2, we may assume that $G[A+\{b,x_1,x_2\}]$ has independent paths Q_1,Q_2 from b to x_1,x_2, respectively.

Suppose B has a component W such that $\{x_1, x_2\} \subseteq N(W)$. Then there exists $w \in V(W)$ such that $G[W + \{b, x_1, x_2\}]$ has independent paths from w to b, x_1, x_2, respectively. So by Lemma 2.4.11, $G[B + S_T]$ has independent paths R_1, R_2, R_3, R_4, R_5 from w to x_1, x_2, b, u_1, u_2, respectively, and internally disjoint from S_T, where $u_1, u_2 \in \{x, x_3, x_4\}$ are distinct. Assume by symmetry that $u_1 \in \{x_3, x_4\}$. If $u_1 = x_3$, then $T \cup bx \cup Q_1 \cup Q_2 \cup R_1 \cup R_2 \cup R_3 \cup (R_4 \cup x_3x)$ is a TK_5 in G' with branch vertices b, w, x, x_1, x_2. If $u_1 = x_4$, then $T \cup (P_4 \cup x_3x) \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup (R_3 \cup bx) \cup (R_4 \cup P_5)$ is a TK_5 in G' with branch vertices t, w, x, x_1, x_2.

Thus, we may assume that no component of B is adjacent to both x_1 and x_2. Since G is 5-connected, we may assume by symmetry that W is a component of B such that $N(x_2) \cap V(W) \neq \emptyset$ and $N(x_1) \cap V(W) = \emptyset$. Let $w \in V(W)$. Since G is 5-connected, $G[W + S_T - x_1]$ has independent paths R_1, R_2, R_3, R_4, R_5 from w to x, x_2, x_3, x_4, b, respectively. Since $N(x_1) \cap B \neq \emptyset$, B has a component B_x such that $N(x_1) \cap V(B_x) \neq \emptyset$. Then $N(x_2) \cap V(B_x) = \emptyset$. Since G is 5-connected, $G[B_x + \{x_1, x_4\}]$ contains a path R from x_1 to x_4. Now $T \cup bx \cup Q_1 \cup Q_2 \cup R_2 \cup (R_3 \cup x_3x) \cup R_5 \cup (R_4 \cup R)$ is a TK_5 in G' with branch vertices b, w, x, x_1, x_2. ∎

2.8 Proof of Theorem 1.0.1

In this section, we complete the proof of Theorem 1.0.1, using the lemmas we have proved so far. Let G be a 5-connected nonplanar graph. We proceed to find a TK_5 in G. By Lemma 2.4.1, we may assume that

(1) G contains no K_4^-.

Let M denote a maximal connected subgraph of G such that

$$H := G/M \text{ is 5-connected and nonplanar, and contains no } K_4^-.$$

Note that $|V(M)| = 1$ (i.e., $H = G$) is possible. Let x denote the vertex of H resulting from the contraction of M. Then, for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, one of the following holds:

$$H/T \text{ contains } K_4^-, \text{ or } H/T \text{ is planar, or } H/T \text{ is not 5-connected.}$$

For convenience, we will use x_T to denote the vertex of H/T resulting from the contraction of T. We may assume that

(2) for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, if F is a TK_5 in H/T then x_T is a branch vertex of F.

For, suppose that F is a TK_5 in H/T in which x_T is not a branch vertex. If $x_T \notin V(F)$ then F is also TK_5 in G. So assume $x_T \in V(T)$. Let $u, v \in V(F)$ such that $x_T u, x_T v \in E(F)$. Since M is connected, $G[M + \{u, v\}]$ has a path P from u to v. Thus, $(F - x) \cup P$ is a TK_5 in G. So we may assume (2).

Suppose there exists $T \subseteq V(H)$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, such that H/T is 5-connected and planar. Then by Lemma 2.4.9, $H - T$ contains K_4^-, contradicting (1). So

(3) for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, if H/T is 5-connected then H/T is nonplanar.

We now show that

(4) if $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$ and if $x_1, x_2, x_3 \in N_{H/T}(x_T)$ such that $H/T - \{x_T v : v \notin \{u_1, u_2, x_1, x_2, x_3\}\}$ contains TK_5 for every choice of distinct $u_1, u_2 \in N_{H/T}(x_T) - \{x_1, x_2, x_3\}$, then G contains TK_5.

To prove (4), let $A = N_G(M \cup T) = N_{H/T}(x_T)$. Consider the subgraph $G[M \cup T + A]$. Since $M \cup T$ is connected, there is a vertex $v \in V(M \cup T)$ such that $G[M \cup T + \{x_1, x_2, x_3\}]$

has independent paths from v to x_1, x_2, x_3, respectively. Since G is 5-connected, $G[M \cup T + A]$ has five independent paths from v to A with only v in common and internally disjoint from A. Hence, by Lemma 2.4.11, there exist distinct $u_1, u_2 \in A - \{x_1, x_2, x_3\}$ such that $G[M \cup T + A]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from v to x_1, x_2, x_3, u_1, u_2, respectively, and internally disjoint from A. Now suppose F is a TK_5 in $H/T - \{x_T v : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$. Then $F - x_T$ and the four paths among P_1, P_2, P_3, P_4, P_5 corresponding to the four edges at x_T in F form a TK_5 in G. Hence, we may assume (4).

By (3), we have two cases: for some $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, H/T is 5-connected and nonplanar but contains K_4^-; or for every $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

Case 1. There exists $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$ such that H/T is 5-connected and nonplanar, and H/T contains K_4^-.

Let $K \subseteq H/T$ such that $K \cong K_4^-$, and let $V(K) = \{x_1, x_2, y_1, y_2\}$ with $y_1 y_2 \notin E(H)$. By (1), $x_T \in V(K)$.

Subcase 1.1. x_T has degree 2 in K.

Then we may assume that the notation is chosen so that $x_T = y_2$. By Lemma 2.4.2, one of the following holds:

(i) H/T contains a TK_5 in which x_T is not a branch vertex.

(ii) $H/T - x_T$ contains K_4^-.

(iii) H/T has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_T, a_1, a_2, a_3, a_4\}$, and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_1$ and the 4-cycle $b_1 b_2 b_3 b_4 b_1$ by adding x_T and the edges $x_T b_i$ for $i \in [4]$.

(iv) For $w_1, w_2, w_3 \in N_{H/T}(x_T) - \{x_1, x_2\}$, $H/T - \{x_T v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.

Note that (i) does not occur because of (2), and (ii) does not occur because of (1).

Now suppose (iii) occurs. First, assume $|V(G_1)| \geq 7$. Then by Lemma 2.4.3, for any $u_1, u_2 \in N(x_T) - \{b_1, b_2, b_3\}$, $H/T - \{x_T v : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$ contains TK_5. Hence, by (4) (with x_i as b_i for $i \in [3]$), G contains TK_5. So we may assume that $|V(G_1)| = 6$, and let $v \in V(G_1 - G_2)$. By (1), $a_i a_{i+1} \notin E(G)$ for $i \in [4]$, where $a_5 = a_1$. Hence, since G is 5-connected, $a_1 a_3, a_2 a_4 \in E(G)$. Now $(H - x_T) - \{a_1 v, a_1 b_4, a_4 v, a_4 b_4\}$ is a TK_5 with branch vertices a_2, a_3, b_1, b_2, b_3, contradicting (2).

Finally, suppose (iv) holds. Then, by (4) (with w_1, w_2, w_3 as x_3, u_1, u_2, respectively), we see that G contains TK_5.

Subcase 1.2. x_T has degree 3 in K.

Then we may assume that the notation is chosen so that $x_T = x_1$. By Lemma 2.4.4, one of the following holds:

(i) H/T contains a TK_5 in which x_T is not a branch vertex.

(ii) $H/T - x_T$ contains K_4^-, or H/T contains a K_4^- in which x_T is of degree 2.

(iii) x_2, y_1, y_2 may be chosen so that for any distinct $z_0, z_1 \in N_{H/T}(x_T) - \{x_2, y_1, y_2\}$, $H/T - \{x_T v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$ contains TK_5.

By (2), (i) does not occur. If (ii) holds then, by (1), H/T contains K_4^- in which x_T is of degree 2; and we are back in Subcase 1.1. If (iii) holds then G contains TK_5 by (4).

Case 2. H/T is not 5-connected for each $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$.

Let $\mathcal{Q}_x$ denote the set of all quadruples (T, S_T, A, B), such that

- $T \subseteq V(H)$, $x \in V(T)$, and $T \cong K_2$ or $T \cong K_3$,
- S_T is a cut in H with $V(T) \subseteq S_T$, A is a nonempty union of components of $H - S_T$, and $B = H - S_T - A \neq \emptyset$,
- if $T \cong K_3$ then $5 \leq |S_T| \leq 6$, and

- if $T \cong K_2$ then $|S_T| = 5$, $|V(A)| \geq 2$, and $|V(B)| \geq 2$.

Among all the quadruples in $\mathcal{Q}_x$, we select (T, S_T, A, B) such that $|V(A)|$ is minimum.

Since $K_4^- \not\subseteq H$, $T \cong K_3$ (by Lemma 2.6.3) and there exists $a \in V(A)$ such that $ax \in E(H)$ (by Lemma 2.6.5 and by (2) and (4)). By Lemma 2.6.2, there exists $T' \subseteq H$ such that $x \in V(T')$ and $T' \cong K_2$ or $T' \cong K_3$, and there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$. Again since $K_4^- \not\subseteq H$, $T' \cong K_3$ by Lemma 2.6.4 and by (2) and (4).

We may assume, without loss of generality, that $T \cap C = \emptyset$. Hence, by Lemma 2.7.1 and by (2) and (4), $A \cap C = \emptyset$ (since $K_4^- \not\subseteq H$). We may assume $B \cap C \neq \emptyset$; for otherwise, $|V(A)| \leq |V(C)| = |V(C) \cap S_T| \leq 3$ and, by Lemma 2.6.1, H contains K_4^-, a contradiction.

We may assume that $|V(T') \cap S_T| = 2$ for any choice of $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$; otherwise, by Lemmas 2.7.3 and 2.7.4, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subseteq H$. Hence, since $K_4^- \not\subseteq H$, we have $A \cap D \neq \emptyset$ by Lemma 2.6.1.

Note that $|S_T| = |S_{T'}| = 6$; for otherwise, by Lemma 2.4.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subseteq H$. We claim that $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. First, note that $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$; otherwise, $(T', (S_T \cup S_{T'}) - V(B \cup C), A \cap D, G[B \cup C]) \in \mathcal{Q}_x$ and $1 \leq |V(A \cap D)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) with $|V(A)|$ minimum. Since H is 5-connected and $B \cap C \neq \emptyset$, $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$. So the claim follows from the fact that $|(S_T \cup S_{T'}) - V(B \cup C)| + |(S_T \cup S_{T'}) - V(A \cup D) = |S_T| + |S_{T'}| = 12$.

If $S_T \cap V(C) = \emptyset$ for some choice $(T', S_{T'}, C, D)$ then $|S_{T'} \cap V(A)| = 1$ as $|S_{T'}| = 6$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$; so by Lemma 2.7.5, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subseteq H$.

Hence, we may assume that

$$S_T \cap V(C) \neq \emptyset$$

for any choice of $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Then $2 \leq |S_T \cap S_{T'}| \leq 4$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$.

Suppose $|S_T \cap S_{T'}| = 4$. Then $|S_{T'} \cap V(B)| = 0$ and $|S_T \cap V(C)| = 1$, as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. Since $|S_T| = |S_{T'}| = 6$, $|S_T \cap V(D)| = 1$ and $|S_{T'} \cap V(A)| = 2$. Hence, $B \cap D \neq \emptyset$ (since $|V(D)| \geq V(A)|$). So $S_T - V(C)$ is a 5-cut in H and $V(T) \subseteq S_T - V(C)$. Note $|V(B \cap D)| \geq 2$; for otherwise, since H is 5-connected, $H[T \cup (B \cap D)]$ contains K_4^-, a contradiction. Hence, by Lemma 2.4.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subseteq H$.

Now assume $|S_T \cap S_{T'}| = 3$. Then, $|S_{T'} \cap V(B)| \leq 1$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$ and $|S_T \cap V(C)| > 0$. Suppose $|S_{T'} \cap V(B)| = 0$. Then $|S_{T'} \cap V(A)| = 3$ as $|S_{T'}| = 6$. So $|S_T \cap V(D)| = 1$ since $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$. Thus, since H is 5-connected, $B \cap D = \emptyset$. However, this implies that $|V(D)| < |V(A)|$, a contradiction. So $|S_{T'} \cap V(B)| = 1$. Then $|S_{T'} \cap V(A)| = 2$ as $|S_{T'}| = 6$, and $|S_T \cap V(C)| = 1$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. Let $q \in S_{T'} \cap V(A - T')$, $S' := (S_{T'} - \{q\}) \cup (S_T \cap V(C))$, $C' := B \cap C$, and $D' = G[D + q]$. Then $(T', S', C', D') \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$ and $T \cap C' = \emptyset$, However, $S_T \cap V(C') = \emptyset$, a contradiction.

Finally, assume $|S_T \cap S_{T'}| = 2$. Suppose $|S_T \cap V(C)| \geq 2$. Then $|S_{T'} \cap V(B)| \leq 1$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$), and $|S_{T'} \cap V(A)| \geq 3$ (as $|S_{T'}| = 6$). So $B \cap D \neq \emptyset$ as $|V(D)| \geq |V(A)|$. Hence, $(S_T \cup S_{T'}) - V(A \cup C)$ is a 5-cut in H and contains $V(T)$. If $|V(B \cap D)| = 1$ then, since H is 5-connected, $H[T \cup (B \cap D)]$ contains K_4^-, a contradiction. So $|V(B \cap D)| \geq 2$. Then, by Lemma 2.4.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subseteq H$. Therefore, we may assume $|S_T \cap V(C)| = 1$. Hence, $|S_T \cap V(D)| = 3$ (as $|S_T| = 6$), $|S_{T'} \cap V(B)| = 2$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$), and $|S_{T'} \cap V(A)| = 2$ (as $|S_{T'}| = 6$). Let $q \in S_{T'} \cap V(A - T')$, $S' := (S_{T'} - \{q\}) \cup (S_T \cap V(C))$, $C' := B \cap C$, and $D' = G[D + q]$. Then $(T', S', C', D') \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$ and $T \cap C' = \emptyset$, However, $S_T \cap V(C') = \emptyset$, a contradiction. ∎

CHAPTER 3

SUBDIVISIONS OF CLIQUES IN C_4-FREE GRAPHS

In this chapter, we focus on subdivisions of large cliques in C_4-free graphs. Mader conjectured that every C_4-free graph with average degree d contains TK_l with $l = \Omega(d)$. Komlós and Szemerédi reduced the problem to expanders and proved Mader's conjecture for n-vertex expanders with average degree $d < \exp(\log^{1/8} n)$. In this chapter, we show that Mader's conjecture is true for n-vertex expanders with average degree $d < n^{3/10}$, which improves Komlós and Szemerédi's quasi-polynomial bound to a polynomial bound. As a consequence, we show that every C_4-free graph with average degree d contains a TK_l with $l = \Omega(d/(\log d)^c)$ for any $c > 3/2$. Independently, Liu and Montgomery resolves Mader's conjecture using the expander method very recently.

3.1 Introduction

We are interested in finding TK_l in graphs, with l large. This is equivalent to the problem for finding $\binom{l}{2}$ internally vertex-disjoint paths between prescribed pairs of vertices. Thus, a result in Robertson and Seymour's graph minors project [44] gives a polynomial time algorithm that determines whether or not a given graph contains a TG. Grohe and Marx [45] proved a structure theorem for graphs containing no TG. However, it is not clear how this structure theorem can be applied to deal with problems in this dissertation.

Bollobás and Thomason [46], and independently Komlós and Szemerédi [47, 48] proved that every graph with average degree d contains TK_l with $l = \Omega(\sqrt{d})$, answering a question of Mader [49] and Erdős and Hajnal [4]. The bound is best possible as the disjoint union of $K_{d,d}$'s contains no TK_l with $l \geq \sqrt{8d}$, see [50]. The proof in [46] is based on the linkage method developed by Robertson and Seymour [44], and the proof in [47, 48] uses the expander method. Mader [11] proved that if G is an n-vertex graph with $3n - 5$ edges

($n \geq 3$) then G contains TK_5, and he [51] made the following

Conjecture 3.1.1 *Every C_4-free graph of average degree d contains TK_l with $l = \Omega(d)$.*

Kühn and Osthus [31] proved that for large r if the C_4-free graph G has girth at least 15 and minimum degree r, then G has a TK_{r+1}. In [52], they show that one can find TK_l with $l = \Omega(d/\log^{12} d)$, in any C_4-free graph with average degree d. By extending the ideas in [47, 48], Balogh, Liu and Sharifzadeh [53] recently proved that every C_6-free graph of average degree d contains a TK_l with $l = \Omega(d)$.

In [47], Komlós and Szemerédi prove Conjecture 3.1.1 for very sparse expander graphs. Let G be a graph. We use $d(G), \delta(G), \Delta(G)$ and $n(G)$ to denote the average degree, the minimum degree, the maximum degree and the number of vertices of G, respectively. For $\kappa > 1$, $\epsilon_1 > 0$ and $t > 0$, let

$$\epsilon(x) = \epsilon(x, \epsilon_1, t) = \begin{cases} 0 & \text{if } x < t/5 \\ \epsilon_1/\log^{\kappa}(15x/t) & \text{if } x \geq t/5 \end{cases}$$

A graph G is an (ϵ_1, t)-expander if $|N(X)\backslash X| \geq \epsilon(|X|)|X|$ for all $X \subseteq V$ with $t/2 \geq |X| \geq |V|/2$. Komlós and Szemerédi [47] proved that for $0 < \epsilon_1 < 1$, if G is an (ϵ_1, d)-expander on n vertices with average degree d and if $d/2 \leq \delta(G) \leq \Delta(G) \leq 72d^2$ and $\log n \geq (\log d)^{\alpha}$, where $\alpha > 7$, then G contains TK_l with $l = \Omega(d)$. In this dissertation, we prove Theorem 1.0.2, which improves the bound $\log n \geq (\log d)^{\alpha}$ to $n \geq d^c$ for any constant $c > 10/3$.

As a consequence of Theorem 1.0.2, we can improve the bound in [52].

Theorem 3.1.2 *For every $\kappa > 1$, there exists $d_0 > 0$ such that every C_4-free graph G of average degree $d > d_0$ contains a TK_l with $l = \Omega(d/(\log d)^{3\kappa/2})$.*

Moreover, Balogh, Liu and Sharifzadeh in [53] proved Conjecture 3.1.1 for very dense graphs (i.e. if G is a C_4-free graph on n vertices with $\Theta(n^{3/2})$ edges), motivated by a result of Alon, Krivelevich and Sudakov [54]. In this paper, we prove the following, which establishes an approximate result when the average degree is close to $\sqrt{n}$.

Theorem 3.1.3 *Let G be a C_4-free bipartite graph on n vertices with average degree d. Suppose $n = d^2 w(d)$ where w is an increasing function with $w(d) = o(d)$. Suppose there exists a function $f : \mathbb{R}^+ \to [1, +\infty)$ such that $w(cx) \leq f(c)w(x)$ for any $c, x \in \mathbb{R}^+$. Then G contains a TK_l with $l = \Omega(d/w(d))$.*

We remark that Theorem 3.1.3 generalizes Balogh, Liu and Sharifzadeh's result by setting $w(x) = 1$ for $x \in \mathbb{R}^+$. Moreover, Theorem 3.1.3 improves the lower bound in Theorem 3.1.2 when $d^2 \leq n \leq d^2 \log^{3/2} d$.

Independently, Liu and Montgomery [55] resolves Conjecture 3.1.1 in a strong sense using the expander method very recently. More specifically, they show that given any integers $s, t \geq 2$, there exists some $c = c(s,t) > 0$ such that any $K_{s,t}$-free graph with average degree d contains a subdivision of a clique with at least $cd^{\frac{s}{2(s-1)}}$ vertices.

The organization of this paper is as follows. In Section 3.2, we introduce some results and lemmas that will be useful in our proofs later. We adopt the dependent random choice technique in Section 3.3, and prove a weaker version of Theorem 1.0.2. Then we modify this proof by a random blow-up trick to prove Theorem 1.0.2. In Section 3.4, we show Theorem 3.1.2 and Theorem 3.1.3. Finally, we conclude in Section 3.5.

3.2 Previous results and useful lemmas

In this section, we list previous results as lemmas to be used later in this paper. Komlós and Szemerédi [48] showed that every graph G contains a robust (ϵ_1, t)-expander that is almost as dense as G.

Lemma 3.2.1 *Let $t > 0$, and choose $\epsilon_1 > 0$ sufficiently small (independent of t) so that $\epsilon = \epsilon(x)$ defined above satisfies $\int_1^\infty \epsilon(x)/x \, dx < 1/8$. Then every graph G has a subgraph H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$, which is an (ϵ_1, t)-expander. Moreover, H has* *the following* ***robustness*** *property: For every $X \subseteq V(H)$, if $|X| < \frac{d(H)}{4\Delta(H)} n(H)\epsilon(n(H))$, then there is a subset $Y \subseteq V(H) - X$ of size $|Y| > n(H) - \frac{2\Delta(H)|X|}{d(H)\epsilon(n(H))}$ such that $H[Y]$ is still an (ϵ_1, t)-expander.*

The following result is Corollary 2.3 in [48] which says that every (ϵ_1, t)-expander graph has "small diameter" and this property is robust with respect to vertex removals.

Lemma 3.2.2 *If G is an n-vertex (ϵ_1, t)-expander, then any two vertex sets, each of size x ($x \geq t$), are of distance at most $(\epsilon_1/2)\log^{1+\kappa}(15n/t)$ and this remains true even after deleting $x\epsilon(x)/4$ arbitrary vertices from G.*

We need a result on reducing the maximum degree of expanders, which is Lemma 2.4 from [53].

Lemma 3.2.3 *Let $0 < \epsilon_1 < 1$ and $\epsilon_2 > 0$. Let G be an n-vertex bipartite C_4-free $(\epsilon_1, \epsilon_2 d^2)$-expander with average degree d and $\delta(G) \geq d/2$. Then either G has TK_l with $l = \Omega(d)$, or G has a C_4-free subgraph G' with average degree $d(G') \geq d/2$ and minimum degree $\delta(G') \geq d(G')/4$ that is $(\epsilon_1/8, 4\epsilon_2 d^2)$-expander. Furthermore, G' has at least $n/2$ vertices and $\Delta(G') \leq d(G')\log^8(|V(G')|/d(G')^2)$.*

3.3 Dependent random choice

In this section, we use the dependent random choice technique to prove Theorem 1.0.2. First, we need some notations. Let G be a graph. For a subset $S \subseteq V(G)$ and an integer $i \geq 1$, let $N_i(S)$ be the ith common neighborhood of S, i.e. $v \in N_i(S)$ if and only if the distance from v to each vertex in S is i. Note that $N(S)$ may not be equal to $N_i(S)$.

Lemma 3.3.1 *Let H be a C_4-free bipartite graph with partitions $A \cup B$. Let $L \subseteq A$ with $|L| = l \geq 2$, and let $\Delta \geq l$. Suppose that each vertex in B has degree at most Δ in H, and $|N_2(S)| \geq 10l\Delta$ for each $S \in \binom{L}{2}$. Then H contains a TK_l, in which every path corresponding to an edge of the K_l has length 4.*

Proof. We show that H has a TK_l with the vertices in L as its branch vertices. We will find internally disjoint paths of length 4, one for each pair of branch vertices. So we list the pairs $\binom{L}{2}$ as $S_1, \ldots, S_q$ where $q = \binom{l}{2}$. For $i \in [q]$ and $v \in N_2(S_i)$, let $B_i(v) := N(v) \cap N(S_i)$. Since H is C_4-free, $|N_1(S_i)| \leq 1$ for $i \in [q]$ and, because $|S_i| = 2$, $|B_i(v)| \leq 2$ for $i \in [q]$ and $v \in N_2(S_i)$.

We find the paths P_i connecting the vertices in S_i, in the order $i = 1, 2, \ldots, q$, such that P_i uses a vertex from $N_2(S_i)$. Since $|N_2(S_1)| \geq 10l\Delta$, there is a path P_1 of length 4 using one vertex from $N_2(S_1)$ and two vertices in $N(S_1)$. Now suppose we have found internally vertex-disjoint paths $P_1, \ldots, P_s$ of length 4 such that for $i \in [s]$, P_i connects the vertices in S_i and uses one vertex from $N_2(S_i)$. Clearly, if $v \in N_2(S_i) \cap V(P_i)$ then P_i uses two vertices in $N(S_i) \cap N(v)$.

To find P_{s+1}, it suffices to show that there exists $v \in N_2(S_{s+1}) - L$ such that $|B_{s+1}(v)| = 2$ and $B_{s+1}(v)$ is disjoint from $\bigcup_{i \in [s]} V(P_i)$; for, in this case, we simply let P_{s+1} be the path using v and the vertices in $B_{s+1}(v) \cup S_{s+1}$. Hence, we count the vertices $v \in N_2(S_{s+1}) - L$ not satisfying this condition, and there are three types.

The first type consists of those vertices $v \in N_2(S_{s+1})$ with $|B_{s+1}(v)| = 1$. There are at most $\Delta - 2$ such vertices, since for any $v \in N_2(S_{s+1})$ with $|B_{s+1}(v)| = 1$, $B_{s+1}(v) \subseteq N_1(S_{s+1})$ and $|N_1(S_{s+1})| = 1$.

The second type consists of those vertices $v \in N_2(S_{s+1})$ used by paths P_i for $i \in [s]$. There are at most $\binom{l}{2} \leq l^2/2$ such vertices, as each P_i uses just one vertex from $A - S_i$.

The third type consists of the vertices $v \in N_2(S_{s+1})$ with $|B_{s+1}(v)| = 2$ and $B_{s+1}(v) \cap V(\cup_{i \in [s]} P_i) \neq \emptyset$. Since each vertex in B has degree at most Δ, $V(P_i) \cap B$ intersects $B_{s+1}(v)$ for at most 2Δ vertices $v \in N_2(S_{s+1})$. For $x \in L - S_{s+1}$, since G is C_4-free, x can be adjacent to $B_{s+1}(v)$ of at most two vertices $v \in N_2(S_{s+1})$. Note that $N_2(S) - L$ has at least $10l\Delta - \Delta - l - l^2/2 \geq 8l\Delta$ that are not of the first two types. Thus, there are at least $8l\Delta - (2l) \cdot (2\Delta) - 2(l-2) \cdot \Delta \geq 2l\Delta$ vertices in $N_2(S_{s+1})$ that are not of these three types. ∎

The next lemma is proved using a dependent random choice argument.

Lemma 3.3.2 *Let $H = (A \cup B, E)$ be a C_4-free bipartite graph on $n(H)$ vertices with average degree $d(H)$. Suppose $|A| \geq |B|$ and $|N_2(v)| \leq M$ for each $v \in A$. Moreover, suppose $\sum_{v \in A} |N_2(v)| \geq c_0 n(H) d(H)^2$ for some constant $c_0 > 0$. If there exist positive integers a, m, t such that*

$$M\Big(\frac{2m}{n(H)}\Big)^t \leq 1 \quad \textit{and} \quad c_0^t d(H)^{2t} n(H)^{1-t} \geq 2a$$

then there exists $U \subseteq A$ with at least a vertices such that for every two vertices x, y in U, $|N_2(\{x, y\})| \geq m$.

Proof. Since $|A| \geq |B|$, $n(H) \geq |A| \geq n(H)/2$. Let $t \geq 1$ be an integer. Choose t vertices from A uniformly at random with repetition, and let T denote the resulting multiset. For convenience, let $W := N_2(T)$; then $W \subseteq A$ and

$$\begin{aligned}
\mathbb{E}[|W|] &= \sum_{v \in A} \mathbb{P}(v \in N_2(T)) = \sum_{v \in A} \mathbb{P}\Big(\bigwedge_{t_0 \in T} v \in N_2(t_0)\Big) = \sum_{v \in A} \mathbb{P}\Big(\bigwedge_{t_0 \in T} t_0 \in N_2(v)\Big) \\
&= \sum_{v \in A} \prod_{t_0 \in T} \mathbb{P}(t_0 \in N_2(v)) = \sum_{v \in A} \mathbb{P}(t_0 \in N_2(v))^t = \sum_{v \in A} \Big(\frac{|N_2(v)|}{|A|}\Big)^t.
\end{aligned}$$

So by applying the Cauchy-Schwarz inequality and the fact $|A| \leq n$, we have

$$\mathbb{E}[|W|] \geq |A|^{1-t}\Big(\frac{\sum_{v \in A} |N_2(v)|}{|A|}\Big)^t \geq |A|^{1-2t}\Big(c_0 n(H) d(H)^2\Big)^t \geq c_0^t d(H)^{2t} n(H)^{1-t}.$$

Let $Y = |\{S \in \binom{W}{2} : |N_2(S)| \leq m\}|$. The probability that a set $S \in \binom{W}{2}$ satisfies $|N_2(S)| \leq m$ is at most $(m/|A|)^t$, because $t_0 \in T$ implies $t_0 \in N_2(S)$. Thus,

$$\mathbb{E}[Y] \leq \mathbb{E}\left[\binom{|W|}{2}\Big(\frac{m}{|A|}\Big)^t\right] \leq \frac{1}{2}\mathbb{E}[|W|^2]\Big(\frac{m}{n(H)/2}\Big)^t \leq \frac{1}{2}\mathbb{E}[|W|]M\Big(\frac{2m}{n(H)}\Big)^t \leq \frac{1}{2}\mathbb{E}[|W|]$$

Hence, $\mathbb{E}[|W| - Y] \geq \frac{1}{2}\mathbb{E}[|W|] \geq a$ by assumption. Therefore, there exists $U \subseteq A$ with at least a vertices such that for every two vertices x, y in U, $|N_2(\{x, y\})| \geq m$. ∎

Now, we are in a position to prove that Mader's conjecture is true for sparse expander graphs. We first prove Theorem 1.0.2 for $c > 4$, as this proof is short and illustrates some ideas in the more involved proof of Theorem 1.0.2.

Proposition 3.3.3 *Let $0 < \epsilon_1 < 1$ and $\epsilon_2 > 0$. Let G be a C_4-free bipartite $(\epsilon_1, \epsilon_2 d^2)$-expander on n vertices with average degree d and minimum degree $\delta(G) \geq d/2$. Suppose $n \geq d^c$ for some constant $c > 4$. Then G contains TK_l with $l = \Omega(d)$.*

Proof. By Lemma 3.2.3, we may assume that G contains a subgraph H such that $n(H) \geq n/2$, $d(H) \geq d/2$, $\delta(H) \geq d(H)/4$ and $\Delta(H) \leq d(H) \log^8(n(H)/d(H)^2)$. Then $2d = 4e(G)/n \geq 4e(G)/2n(H) \geq 2e(H)/n(H) = d(H)$. Let A, B be the bipartition of H inherited from G, and assume $|A| \geq |B|$. Thus $n(H) \geq |A| \geq n(H)/2$.

Since H is C_4-free, $|N_2(v)| \geq \delta(H)^2$ for $v \in A$. So

$$\sum_{v \in A} |N_2(v)| \geq |A|\delta(H)^2 \geq \frac{1}{32} n(H) d(H)^2.$$

Moreover, for any $v \in A$, $|N_2(v)| \leq \Delta(H)^2 \leq d(H)^2 (\log n(H))^{16}$. Let $t = 1$, $c_0 = \frac{1}{32}$, $m = 10d(H)^2(\log n(H))^8, a = \frac{1}{64}d(H)^2$ and $M = d(H)^2(\log n(H))^{16}$. Then

$$c_0^t d(H)^{2t} n(H)^{1-t} = \frac{1}{32} d(H)^2 \geq 2a$$

and, since $c > 4$ and $n(H) \geq n/2 \geq \frac{1}{2}d^c \geq \frac{1}{2}(d(H)/2)^c$,

$$M\Big(\frac{2m}{n(H)}\Big)^t = \frac{20d(H)^4(\log n(H))^{24}}{n(H)} \leq 1.$$

Hence, by applying Lemma 3.3.2 to H with parameters a, m, t, c_0 above, there exists $U \subseteq A$ with $|U| \geq \frac{1}{64}d(H)^2$ such that for any $x, y \in U$, $|N_2(\{x, y\})| \geq 10d(H)^2 (\log n(H))^8$.

Let L be a subset of U of size $d(H)$. By Lemma 3.3.1 with $l = d(H)$ and $\Delta = d(H)(\log(n(H)))^8 \geq \Delta(H)$, we obtain a $TK_{d(H)}$ with the vertices in L as its branch vertices. ∎

Note that in the proof of Proposition 3.3.3, t is an integer. In order to improve $c > 4$ to $c > 10/3$, we need to consider a fractional version. This is done by blowing up the vertices and edges in the original graph. To make the new graph also C_4-free, we add edges randomly and perform alterations. (This step, see the claim below, uses Chernoff bounds and requires tedious calculations; so we leave the detailed arguments to the appendix.) By showing the correspondence between a topological minor in the blow-up graph and one in the original graph, we can prove Theorem 1.0.2.

Proof of Theorem 1.0.2. By Lemma 3.2.3 we may assume that G has a C_4-free subgraph H with $n(H) \geq n/2$, $d(H) \geq d/2$, $\Delta(H) \leq d(H)\log^8(n(H)/d(H)^2)$, and $\delta(H) \geq d(H)/4$. Let A, B be a bipartition of H such that $|A| \geq |B|$, and thus $n(H) \geq |A| \geq n(H)/2$.

Let $c > 10/3$ be fixed. We will find a TK_l in H with $l = \Omega(d)$. Let $\epsilon > 0$ be sufficiently small. By Proposition 3.3.3, we may assume that $n(H) \leq d(H)^{4+(2\epsilon/(1-2\epsilon))}$. Hence, $n(H) = d(H)^c$ with $10/3 < c \leq 4 + (2\epsilon/(1-2\epsilon))$.

We now construct a new graph J from H. Let $s = (3c-10)/(2c-6) - 2\epsilon$ and $r = (1-\frac{\epsilon}{4})/(3-2s)$. So $0 < s \leq 1-\epsilon$ and $(1-\frac{\epsilon}{4})/3 \leq r < 1-\frac{\epsilon}{4}$. Let $p = \lceil d(H)^r \rceil$ and $q = \lceil p^s \rceil$. The vertex set of J is the disjoint union of $\{x_1, x_2, ..., x_p\}$ for $x \in V(H)$. For each $xy \in E(H)$ and for all $i \in [p]$ and $j \in [p]$, let $x_i y_j \in E(J)$ with probability q/p^2. Clearly, J is a bipartite graph with partition classes A', B', where A' (respectively, B') is the union of $\{x_1, \ldots, x_p\}$ for $x \in A$ (respectively, $x \in B$).

Next, we obtain a C_4-free graph J' from J by removing an edge from each C_4 in J. We have the following claim, whose proof is given in the appendix.

Claim. With probability $1 - o(1)$, the following properties hold:

(i) $\frac{q}{2p}\delta(H) \leq \delta(J) \leq \Delta(J) \leq \frac{2q}{p}\Delta(H)$.

(ii) $\frac{q}{2p}d(H) \leq d(J) \leq \frac{3q}{2p}d(H)$.

(iii) $\frac{q}{3p}d(H) \leq d(J') \leq \frac{3q}{2p}d(H)$, and $\delta(J') \geq \frac{q}{4p}\delta(H)$.

Note that J' is also a bipartite graph with partition classes A', B'. Since $|A| \geq |B|$, $n(J') \geq |A'| \geq n(J')/2$. Since $\delta(H) \geq d(H)/4$, it follows from (iii) that

$$\delta(J') \geq \frac{q}{4p}\delta(H) \geq \frac{q}{16p}d(H) \geq \frac{q}{16p}\frac{2p}{3q}d(J') = d(J')/24.$$

Since J' is C_4-free, $|N_2(v)| \geq \delta(J')^2$ for $v \in A'$; so

$$\sum_{v \in A'} |N_2(v)| \geq |A'|\delta(J')^2 \geq \frac{n(J')}{2}(\frac{d(J')}{24})^2 = \frac{1}{1152}n(J')d(J')^2$$

Moreover, by (i), for any $v \in A'$,

$$|N_2(v)| \leq \Delta(J')^2 \leq \frac{4q^2}{p^2}\Delta(H)^2 \leq \frac{4q^2}{p^2}d(H)^2(\log n(H))^{16}.$$

We wish to apply Lemma 3.3.2 to J' with the parameters $a = pd(H)$, $c_0 = \frac{1}{1152}$, $m = 10pd(H)^2(\log n(H))^8$, $M = \frac{4q^2}{p^2}d(H)^2(\log n(H))^{16}$, and $t = 1$. First, note that by (iii),

$$c_0^t d(J')^{2t} n(J')^{1-t} \geq \frac{1}{1152}(\frac{q}{3p}d(H))^2 = 2a\frac{q^2 d(H)}{20736p^3} = 2a\frac{d(H)^{2sr+1-3r}}{20736}.$$

Note that $2sr + 1 - 3r = \frac{\epsilon}{4}$. Hence,

$$c_0^t d(J')^{2t} n(J')^{1-t} \geq 2a\frac{d(H)^{\epsilon/4}}{20736} \geq 2a.$$

Next, we have

$$M\Big(\frac{2m}{n(J')}\Big)^t = M\frac{2m}{n(J')} = 80c^{24}d(H)^{2sr+4-2r-c}(\log d(H))^{24}.$$

Note that

$$2sr+4-2r-c = \frac{(2c-6-\epsilon/2)s+(10-3c+\epsilon/2)}{3-2s} = \frac{1/2-2(2c-6)-\frac{1}{2}(\frac{3c-10}{2c-6}-2\epsilon)}{3-2s}\epsilon.$$

Since $0 < s \leq 1-\epsilon$, we have $1 < 3-2s < 3$ and $\frac{3c-10}{2c-6} - 2\epsilon > 0$. Moreover, since $c > 10/3$, we deduce that

$$2sr+4-2r-c < \frac{1/2-2(2\cdot\frac{10}{3}-6)}{3}\epsilon = -\frac{5}{18}\epsilon.$$

Hence,

$$M\Big(\frac{2m}{n(J')}\Big)^t < 80c^{24}d(H)^{-\frac{5}{18}\epsilon}(\log d(H))^{24} \leq 1.$$

Hence, by Lemma 3.3.2, there exists a set $U' \subseteq A'$ of size $pd(H)$ such that any two vertices in U' have at least $10pd(H)^2(\log n(H))^8$ common second neighbors in J'. Let $U = \{x \in V(H) : x_i \in U' \text{ for some } i \in [p]\ \}$. Then $|U| \geq d(H)$ and any two vertices in U have at least $10d(H)^2(\log n(H))^8$ common second neighbors in H. We apply Lemma 3.3.1 to H with $l = d(H)$ and $\Delta = d(H)(\log(n(H)))^8 \geq \Delta(H)$ and obtains a $TK_{d(H)}$ with the vertices in U as its branch vertices. ∎

3.4 A new lower bound on maximum clique subdivisions

In this section, we prove Theorem 3.1.2. First, in n-vertex expanders whose average degree is at least n^α for some $0 < \alpha < 1/2$, we can use the second neighborhood of a vertex to find a TK_l with l large.

Proposition 3.4.1 *Let $\kappa > 1$. Let H be a bipartite C_4-free $(\epsilon_1, \epsilon_2 d^2)$-expander on n vertices with $d(H) \geq d$ and $\delta(H) \geq d/2$. Suppose $n \leq d^\tau$ for some $\tau > 2$. Then H has a TK_l with $l = \Omega(d/(\log d)^{3\kappa/2})$.*

Proof. Let $c = \frac{\epsilon_1}{16\tau^{3\kappa/2}}$ and $l = \frac{cd}{(\log d)^{3\kappa/2}}$. We find a TK_l in H by choosing vertices $v_1, v_2, ..., v_l$ (in that order) as branch vertices and choosing, for each $i \in [l]$, $S_1(v_i) \subseteq N(v_i)$

and $S_2(v_i) \subseteq N(S_1(v_i)) - \{v_i\}$ such that

(i) $|S_1(v_i)| = d/4$ for $i \in [l]$ and $S_1(v_i) \cap S_1(v_j) = \emptyset$ distinct $i, j \in [l]$,

(ii) for each $w \in S_1(v_i)$, $|N(w) \cap S_2(v_i)| = d/4$, and

(iii) $S_2(v_i)$ is disjoint from $\cup_{j\in[l]} S_1(v_j)$ and $v_i \notin B_2(v_j) := \{v_j\} \cup S_1(v_j) \cup S_2(v_j)$ for $i \neq j$.

We claim that such $v_1, ..., v_l$ exist. Clearly, we can choose an arbitrary vertex as v_1, choose $d/4$ neighbors of v_1 to form $S_1(v_1)$, and for each $w \in S_1(v_1)$ we choose a set A_w of $d/4$ neighbors of w other than v_1 and let $S_2(v_1) = \cup_{w\in S_1(v_1)} A_w$. Since G is C_4-free, $|S_2(v_1)| = d^2/16$. Let $B_1(v_1) := \{v_1\} \cup S_1(v_1)$ and $B_2(v_1) := \{v_1\} \cup S_1(v_1) \cup S_2(v_1)$.

Suppose we have found $v_j, S_1(v_j), S_2(v_j), B_1(v_j), B_2(v_j)$ for $j = 1, \ldots, i$. We show how to find $v_{i+1}, S_1(v_{i+1}), S_2(v_{i+1}), B_1(v_{i+1}), B_2(v_{i+1})$. Since H is C_4-free and $\delta(H) \geq d/2$, $n \geq \delta(H)^2 \geq d^2/4 \gg l(d/4 + 1) \geq |\cup_{j=1}^{i} B_1(v_j)|$. Hence, we may choose $v_{i+1} \in V(H) - \cup_{j=1}^{i} B_1(v_j)$. Again since H is C_4-free, $|N(v_s) \cap N(v_t)| \leq 1$ for distinct $s, t \in [i + 1]$. So v_{i+1} has at least $d/2 - 2l > d/4$ neighbors disjoint from $\cup_{j=1}^{i} B_1(v_j)$, and we may choose $S_1(v_{i+1}) \subseteq N(v_{i+1}) - \cup_{j=1}^{i} B_1(v_j)$ with $|S_1(v_{i+1})| = d/4$. Let $B_1(v_{i+1}) := \{v_{i+1}\} \cup S_1(v_{i+1})$. Since $\delta(H) \geq d/2$, for each $w \in S_1(v_i)$, $|N(w)| \geq d/2$. Since H is C_4-free, $|N(w) \cap B_1(v_j)| \leq 1$ for $j \in [i]$. Thus, $|N(w) - \cup_{j=1}^{i} B_1(v_j)| \geq d/2 - l \gg d/4$. Thus, we may choose $S_2(v_{i+1}) \subseteq N(S_1(v_{i+1})) - \{v_{i+1}\} - \cup_{j=1}^{i} B_1(v_j)$ with $|S_2(v_{i+1})| = d^2/16$. Let $B_2(v_{i+1}) := \{v_{i+1}\} \cup S_1(v_{i+1}) \cup S_2(v_{i+1})$. Note that $|B_2(v_i)| = d^2/16 + d/4 + 1$.

Having constructed $v_j, S_1(v_j), S_2(v_j), B_1(v_j), B_2(v_j)$ for $j = 1, \ldots, l$, we can proceed to form a TK_l in H with branch vertices $v_1, \ldots, v_l$. Arbitrarily order all the pairs from $V := \{v_1, \ldots, v_l\}$ as $S_1, \ldots, S_t$, where $t = \binom{l}{2}$. We will find paths P_i between vertices in S_i in order $i = 1, \ldots, l$. We want the paths P_i to be internally disjoint and short (so that removing vertices from the paths has less effect on the connectivity of the remaining graph), and avoid the vertices in $\cup_{v\in L-S_i} B_1(v)$.

First, find a shortest path P_1 in $\cup_{v\in L-S_1} B_1(v)$ between the vertices in S_1. Since H is an $(\epsilon_1, \epsilon_2 d^2)$-expander, $e(P_1) \le \frac{2}{\epsilon_1}\log^{1+\kappa}(n)$. Suppose we have found internally disjoint paths $P_1, \ldots, P_i$ such that for each $j \in [i]$, P_j is a path in $H - ((\cup_{s=1}^{j-1} P_s) \cup (\cup_{v\in L-S_j} B_1(v)) - S_j)$ between the vertices in S_j such that $e(P_j) \le \frac{2}{\epsilon_1}\log^{1+\kappa}(n)$ for $j = 1, \ldots, i$. Then, since $\log n \le \tau \log d$ for some $\tau > 2$,

$$| \cup_{j=1}^{i} V(P_j)| + | \cup_{m=1}^{l} B_1(v_m)| + | \cup_{v\in S_{i+1}} N(S_1(v) \cap (\cup_{j=1}^{i} V(P_j)))|$$

$$\le \binom{l}{2}\frac{2}{\epsilon_1}\log^{1+\kappa}(n) + l + ld/4 + 2ld/4 \le \frac{1}{4}\frac{d^2}{16}\frac{\epsilon_1}{\log^{\kappa}(n)} \le \frac{1}{4}|B_2(v_i)|\epsilon(|B_2(v_i)|).$$

Hence, by Lemma 3.2.2, we can find a path between $B_2(x)$ and $B_2(y)$ of length at most $e(P_{i+1}) \le \frac{2}{\epsilon_1}\log^{1+\kappa}(n)$, where $S_{i+1} = \{x, y\}$, in $H - ((\cup_{s=1}^{i} P_s) \cup (\cup_{v\in L-S_{i+1}} B_1(v)) - S_{i+1})$, which can be extended to a path between x and y. Clearly, $P_1, \ldots, P_t$ form a TK_l in H. ∎

Proof of Theorem 3.1.2. It is well known that G has a bipartite subgraph G' with $d(G') \ge d(G)/2$. Applying Lemma 3.2.1 to G', we obtain a (ϵ_1, t)-expander H with robustness property. Note that H is still C_4-free and bipartite. Moreover, $d(H) \ge d(G')/2 \ge d(G)/4$ and $\delta(H) \ge d(H)/2$.

Let $\tau = 5$. If $n(H) \ge d(H)^{\tau}$, then by Proposition 3.3.3, H contains a TK_l with $l = \Omega(d)$ and so does G. If $n(H) \le d(H)^{\tau}$, then by Proposition 3.4.1, H contains a TK_l with $l = \Omega(d/(\log d)^{3\kappa/2})$ and so does G. ∎

Proof of Theorem 3.1.3. Let G' be an induced subgraph of G that maximizes the average degree $d(G')$. Hence, $n(G') \le n$, $d(G') \ge d$, and $n(G') \le n(G) \le d^2 w(d) \le d(G')^2 w(d(G'))$.

Claim. There exists a subgraph $H \subseteq G'$ such that $V(H) = A \cup B$, $|A| = |B| = n(G')/2$, $d(H) \ge 0.36 d(G')$ and all vertices in B have degree less than $30d(H)$.

Let $X \subseteq V(H)$ be the set of vertices of degree at least $10d(G')$; so $|X| \le n(G')/10$. Let $Y = V(G')\backslash X$. By the choice of G', $d(G'[X]) \le d(G')$, we have $e(G'[X]) \le$

$d(G')|X|/2 \leq e(G')/10$. Take an $\frac{n(G')}{2}$-subset B of Y uniformly at random and define $A = V(G')\backslash B$. Note that $0.9n(G') \leq |Y| \leq n(G')$. Then,

$$\begin{aligned}\mathbb{E}[e(G'[A,B])] &= \frac{(|Y| - n(G')/2)n(G')/2}{\binom{|Y|}{2}} e(G'[Y]) + \frac{n(G')/2}{|Y|} e(G'[X,Y]) \\ &\geq \frac{40}{81} e(G'[Y]) + 0.5e(G'[X,Y]) \\ &\geq 0.4(e(G') - e(G'[X])) \geq 0.36e(G')\end{aligned}$$

Therefore, there exists a choice of A, B such that $e(G'[A,B]) \geq 0.36e(G')$. Let $H = G'[A,B]$ and every vertex in B has degree less than $10d(G') \leq 10d(H)/0.36 \leq 30d(H)$. This completes the proof of Claim.

From now on, we work with the graph H. Notice that

$$n(H) = n(G') \leq d(G')^2 w(d(G')) \leq 9d(H)^2 w(3d(H)).$$

Let w' be a function such that $n(H) = d(H)^2 w'(d(H))$; then $1 \leq w'(d(H)) \leq 9w(3d(H))$. It suffices to show that G has a TK_l with $l = \Omega(d(H)/w'(d(H)))$, since

$$d(H)/w'(d(H)) \geq \frac{1}{9} d(H)/w(3d(H)) \geq \frac{1}{9f(3)} d(H)/w(d(H)).$$

We will apply Lemma 3.3.2 with parameters $t = \lceil \frac{\log d(H) + (1/2)\log w'(d(H))}{\log(32w'(d(H)))} \rceil$, $c_0 = 1/32$, $m = \frac{1}{2048} d(H)^2/w'(d(H))$, $a = \frac{1}{614400} d(H)/w'(d(H))$ and $M = n(H)/2$.

We now check the conditions of Lemma 3.3.2. Note that for any $v \in A$, $|N_2(v)| \leq n(H)/2 = M$. Since H is C_4-free,

$$\sum_{v \in A} |N_2(v)| = \sum_{v \in B} (d(v) - 1)d(v) = \sum_{v \in B} d(v)^2 - \sum_{v \in B} d(v).$$

So by the Cauchy-Schwarz inequality, we have

$$\sum_{v\in A}|N_2(v)| \geq \frac{n(H)}{2}\Big(\frac{\sum_{v\in B} d(v)}{n(H)/2}\Big)^2 - e(H) = \frac{n(H)}{2}(d(H)^2 - d(H))$$
$$\geq \frac{n(H)}{4}d(H)^2 \geq \frac{1}{32}n(H)d(H)^2.$$

Next we note that $M\Big(\frac{2m}{n(H)}\Big)^t \leq 1$ iff $t \geq \frac{\log M}{\log n(H) - \log(2m)}$. The latter is true as

$$\frac{\log M}{\log n(H) - \log(2m)} = \frac{\log d(H) + 1/2 \log w'(d(H)) - 1/2 \log 2}{\log w'(d(H)) + \log 32}.$$

Similarly, $c_0^t d(H)^{2t} n(H)^{1-t} \geq 2a$ iff $t \leq \frac{\log n(H) - \log(2a)}{\log n(H) - 2\log d(H) - \log c_0}$, and the latter holds as it is equal to $\frac{\log d(H) + 2\log w'(d(H)) + \log 30720}{\log w'(d(H)) + \log 32}$.

Hence, by Lemma 3.3.2, there exists a set U of vertices of size $\frac{1}{614400}d(H)/w'(d(H))$ such that any two vertices in U have at least $\frac{1}{2048}d(H)^2/w'(d(H))$ common second neighbors. Therefore, by applying Lemma 3.3.1 to H with $L = U$, $\Delta = 30d(H)$ and $l = \frac{1}{614400}d(H)/w'(d(H))$, we conclude that H has a TK_l. ∎

3.5 Concluding remarks

In [51], Mader posed several conjectures regarding clique subdivisions under additonal girth conditions. For example, he conjectured that if G is a graph with $\delta(G) \geq 4$ and has girth at least 5, then G contains TK_5. (Note that $K_{4,4}$ shows that the girth condition cannot be dropped.) More generally, Mader asked whether G contains TK_{k+1} if $\delta(G) \geq k$ and the girth of G is at least 5.

Appendices

APPENDIX A

PROOF OF CLAIM IN THE PROOF OF THEOREM 3.1.2

In this section, we prove the claim in the proof of Theorem 1.0.2. This proof makes heavy use of the Chernoff bounds given next, whose proof can be found in [56].

Lemma A.0.1 *Suppose $X_1, ..., X_n$ are independent random variables taking values in $\{0, 1\}$. Let X denote their sum and $\mu = \mathbb{E}[X]$ denote the expected value of X. Then for any $0 < \delta \leq 1$,*

$$\mathbb{P}[X \geq (1+\delta)\mu] < e^{-\frac{\delta^2 \mu}{3}}$$

$$\mathbb{P}[X \leq (1-\delta)\mu] < e^{-\frac{\delta^2 \mu}{2}}$$

And for $\delta > 1$,

$$\mathbb{P}[X \geq (1+\delta)\mu] < e^{-\frac{\delta \mu}{3}}$$

We use the notation from the proof of Theorem 1.0.2. For simplicity, we use $n(H) = d(H)^c$, $p = d(H)^r$ and $q = p^s$ (since taking celiing and floor functions do not affect our final results). Recall that $\epsilon > 0$ is a sufficiently small number, $10/3 < c < 4 + 2\epsilon/(1-2\epsilon)$, $s = (3c-10)/(2c-6) - 2\epsilon$, and $r = (1-\epsilon/4)/(3-2s)$. Consequently,

$$0 < s \leq 1 - \epsilon \text{ and } \frac{1-\epsilon/4}{3} < r \leq \frac{1-\epsilon/4}{1+2\epsilon}.$$

Hence, since $1 + sr - r = 1 + (1-\epsilon/4)\frac{s-1}{3-2s}$, we have

$$1 + sr - r > \frac{2}{3} + \frac{\epsilon}{12}. \tag{A.1}$$

Since $4sr - 5r + 2 = 2 + (1 - \epsilon/4)\frac{4s-5}{3-2s}$, we have

$$4sr - 5r + 2 > \frac{1}{3} + \frac{5}{12}\epsilon. \tag{A.2}$$

Moreover, since $3sr - 4r + 1 = 1 + (1 - \epsilon/4)\frac{3s-4}{3-2s}$, we have

$$-\frac{1}{3} + \frac{\epsilon}{3} < 3sr - 4r + 1 \leq -\frac{3}{4}\frac{\epsilon - \epsilon^2}{1 + 2\epsilon} < 0. \tag{A.3}$$

Suppose v is a vertex in J and v_0 is the corresponding vertex in H. Let $v_0u_1, v_0u_2, ...,$ v_0u_k be the edges incident to v in H where $k = d_H(v_0)$. For $i \in [k], j \in [p]$, let u_{ij} be the jth vertex of J corresponding to u_i in H. For $i \in [k], j \in [p]$, define random variables

$$X_{ij}^v = \begin{cases} 1 & \text{if } vu_{ij} \in E(J) \\ 0 & \text{if } vu_{ij} \notin E(J). \end{cases}$$

Let $X^v = \sum_{i\in[k],j\in[p]} X_{ij}^v$, which is the degree of v in J. By linearity of expectations,

$$\mathbb{E}[X^v] = \sum_{i\in[k],j\in[p]} \mathbb{E}[X_{ij}^v] = kp\frac{q}{p^2} = \frac{q}{p}k.$$

(i) With probability $1 - o(1)$, $\frac{q}{2p}\delta(H) \leq \delta(J) \leq \Delta(J) \leq \frac{2q}{p}\Delta(H)$.

Proof. By Lemma A.0.1 and $\delta(H) \geq d(H)/4$,

$$\mathbb{P}[X^v \geq 2\frac{q}{p}\Delta(H)] \leq \mathbb{P}[X^v \geq 2\mathbb{E}[X^v]] < e^{-\mathbb{E}[X^v]/3} = e^{-qk/3p} \leq e^{-qd(H)/12p},$$

and

$$\mathbb{P}[X^v \leq \frac{q}{2p}\delta(H)] \leq \mathbb{P}[X^v \leq \frac{1}{2}\mathbb{E}[X^v]] < e^{-\mathbb{E}[X^v]/8} = e^{-qk/8p} \leq e^{-qd(H)/32p}.$$

By union bound, the probability that there exists $v \in V(J)$ with $X^v < \frac{q}{2p}\delta(H)$ or $X^v > \frac{2q}{p}\Delta(H)$ is less than

$$
\begin{aligned}
&|V(J)|(e^{-qd(H)/12p} + e^{-qd(H)/32p}) \\
&< 2pn(H)\exp(-qd(H)/32p) \\
&= 2\exp((r+c)\log d(H) - \frac{1}{32}d(H)^{sr+1-r}).
\end{aligned}
$$

Hence, by (A.1), the probability that there exists $v \in V(J)$ with $X^v < \frac{q}{2p}\delta(H)$ or $X^v > \frac{2q}{p}\Delta(H)$ is less than $2\exp(6\log d(H) - \frac{1}{32}d(H)^{2/3}) = o(1)$. ∎

(ii) With probability $1 - o(1)$, $\frac{q}{2p}d(H) \leq d(J) \leq \frac{3q}{2p}d(H)$.

Proof. Let $X = \sum_{v \in V(J)} X^v$. Then $X = 2e(J)$ and

$$
\mathbb{E}[X] = \sum_{v \in V(J)} \mathbb{E}[X^v] = \frac{q}{p}\sum_{v \in V(J)} d_H(v_0) = qd(H)n(H)
$$

Moreover, by Lemma A.0.1 and $c > 10/3$

$$
\begin{aligned}
\mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{1}{2}\mathbb{E}[X]] &< 2e^{-\mathbb{E}[X]/8} = 2e^{-\frac{qd(H)n(H)}{8}} \\
&= 2e^{-\frac{d(H)^{1+sr+c}}{8}} \leq 2e^{-\frac{1}{8}d(H)^{13/3}} = o(1).
\end{aligned}
$$

Since $d(J) = X/n(J) = X/pn(H)$, $\frac{q}{2p}d(H) \leq d(J) \leq \frac{3q}{2p}d(H)$ with probability $1 - o(1)$. ∎

(iii) With probability $1 - o(1)$, $\frac{q}{3p}d(H) \leq d(J') \leq \frac{3q}{2p}d(H)$, and $\delta(J') \geq \frac{q}{4p}\delta(H)$.

Proof. To prove (iii) we need to analyze C_4's in J. A C_4 in J is said to be of *type* I (respectively, *type* II) if it corresponds to a path of length 1 (respectively, length 2) in H after identification of blow-up vertices. Let C_I, C_{II} be the numbers of C_4's in J of type I, type II, respectively. Then

$$\mathbb{E}[C_I] = e(H) \cdot \binom{p}{2} \cdot \binom{p}{2} \cdot \left(\frac{q}{p^2}\right)^4 = \frac{q^4 d(H) n(H)}{8p^4}(1+o(1))$$

and

$$\mathbb{E}[C_{II}] = \Big(\sum_{v \in V(H)} \binom{d_H(v)}{2}\Big) \cdot \binom{p}{2} \cdot p \cdot p \cdot \left(\frac{q}{p^2}\right)^4$$

We have the following bounds on $\mathbb{E}[C_{II}]$, where the lower bound follows from the Cauchy-Schwarz inequality and the upper bound follows from the fact $\Delta(H) \leq d(H)\log^8(n(H))$:

$$\frac{q^4 d(H)^2 n(H)}{4p^4}(1+o(1)) \leq \mathbb{E}[C_{II}] \leq \frac{q^4 d(H)^2 n(H) \log^8 n(H)}{4p^4}(1+o(1)).$$

Thus C_4's of type II account for the majority of edges we remove from J to form J'.

By construction, $0 \leq e(J) - e(J') \leq C_I + C_{II}$. So $d(J') \leq d(J) \leq \frac{3q}{2p} d(H)$ by (ii). By Lemma A.0.1, $c > 10/3$ and by (A.3),

$$\begin{aligned}\mathbb{P}[|C_I - \mathbb{E}[C_I]| \geq \frac{1}{2}\mathbb{E}[C_I]] < 2e^{-\mathbb{E}[C_I]/8} &\leq 2\exp\big(-\frac{q^4 d(H) n(H)}{64p^4}\big) \\ &= 2\exp\big(-\frac{d(H)^{4sr-4r+1+c}}{64}\big) = o(1),\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[|C_{II} - \mathbb{E}[C_{II}]| \geq \frac{1}{2}\mathbb{E}[C_{II}]] < 2e^{-\mathbb{E}[C_{II}]/8} &\leq 2\exp\big(-\frac{q^4 d(H)^2 n(H)}{32p^4}\big) \\ &= 2\exp\big(-\frac{d(H)^{4sr-4r+2+c}}{32}\big) = o(1).\end{aligned}$$

Hence, with probability $1 - o(1)$, we remove at most $\frac{3}{2}(\mathbb{E}[C_I] + \mathbb{E}[C_{II}])$ edges from J. Since $d(J) \geq \frac{q}{2p} d(H)$ by (ii), $d(J') \geq \frac{q}{3p} d(H)$ because

$$\frac{3}{2}\frac{q^4 d(H)^2 n(H) \log^8 n(H)}{4p^4}(1+o(1)) = \frac{q}{p} d(H) p n(H) \frac{3q^3 d(H) \log^8 n(H)}{8p^4}(1+o(1)),$$

which is at most

$$\frac{q}{p}d(H)pn(H)\frac{3d(H)^{3sr+1-4r}c^8\log^8 d(H)}{8}(1+o(1)) < \frac{1}{2}\frac{q}{6p}d(H)pn(H),$$

where the last inequality follows from (A.3). So , with probability $1-o(1)$, $d(J') \geq \frac{q}{3p}d(H)$.

We now proceed to prove $\delta(J') \geq \frac{q}{4p}\delta(H)$. For $v \in V(J)$, let v_0 be the corresponding vertex in H. Let C_I^v be the number of C_4's of type I in J containing the vertex v. Let $C_{II,1}^v$ be the number of C_4's of type II in J containing the vertex v such that v_0 is the degree 1 vertex in the path to which C_4 corresponds after identification of the blow-up vertices. Let $C_{II,2}^v$ be the number of C_4's of type II in J containing the vertex v such that v_0 is the degree 2 vertex in the path to which C_4 corresponds after identification of the blow-up vertices. We have the following:

$$\mathbb{E}[C_I^v] = d_H(v_0)\cdot\binom{p}{2}\cdot(p-1)\cdot\left(\frac{q}{p^2}\right)^4 = \frac{q^4 d_H(v_0)}{2p^5}(1+o(1)).$$

$$\mathbb{E}[C_{II,1}^v] \geq d_H(v_0)\cdot\binom{p}{2}\cdot p\cdot\delta(H)\cdot\left(\frac{q}{p^2}\right)^4 = \frac{q^4 d_H(v_0)\delta(H)}{2p^5}(1+o(1)).$$

$$\mathbb{E}[C_{II,1}^v] \leq d_H(v_0)\cdot\binom{p}{2}\cdot p\cdot\Delta(H)\cdot\left(\frac{q}{p^2}\right)^4 = \frac{q^4 d_H(v_0)\Delta(H)}{2p^5}(1+o(1)).$$

$$\mathbb{E}[C_{II,2}^v] = \binom{d_H(v_0)}{2}\cdot p\cdot p\cdot(p-1)\cdot\left(\frac{q}{p^2}\right)^4 = \frac{q^4 d_H(v_0)^2}{2p^5}(1+o(1)).$$

By Lemma A.0.1,

$$\mathbb{P}[|C_I^v - \mathbb{E}[C_I^v]| \geq d(H)\mathbb{E}[C_I^v]] < 2e^{-d(H)\mathbb{E}[C_I^v]/3} \leq 2\exp\left(-\frac{q^4 d(H)^2}{24p^5}\right),$$

$$\mathbb{P}[|C_{II,1}^v - \mathbb{E}[C_{II,1}^v]| \geq \frac{1}{2}\mathbb{E}[C_{II,1}^v]] < 2e^{-\mathbb{E}[C_{II,1}^v]/8} \leq 2\exp\left(-\frac{q^4 d(H)^2}{256p^5}\right),$$

$$\mathbb{P}[|C_{II,2}^v - \mathbb{E}[C_{II,2}^v]| \geq \frac{1}{2}\mathbb{E}[C_{II,2}^v]] < 2e^{-\mathbb{E}[C_{II,2}^v]/8} \leq 2\exp\left(-\frac{q^4 d(H)^2}{256p^5}\right).$$

By the choice of p, q and by (A.2),

$$\begin{aligned}
&pn(H) \cdot (2\exp(-\frac{q^4 d(H)^2}{24p^5}) + 4\exp(-\frac{q^4 d(H)^2}{256p^5})) \\
&\leq 6pn(H)\exp(-\frac{q^4 d(H)^2}{24p^5}) \\
&\leq 6\exp(\log p + \log n(H) - \frac{q^4 d(H)^2}{24p^5}) \\
&\leq 6\exp(r\log d(H) + c\log d(H) - \frac{d(H)^{4sr+2-5r}}{24}) \\
&= o(1).
\end{aligned}$$

By the union bound, with high probability, for all $v \in V(J)$,

$$C_I^v \leq (d(H)+1)\mathbb{E}[C_I^v], C_{II,1}^v \leq \frac{3}{2}\mathbb{E}[C_{II,1}^v], \text{ and } C_{II,2}^v \leq \frac{3}{2}\mathbb{E}[C_{II,2}^v].$$

Therefore, there exists a graph J such that the number of edges being removed for each vertex v of J is at most $(d(H)+1)\mathbb{E}[C_I^v] + \frac{3}{2}\mathbb{E}[C_{II,1}^v] + \frac{3}{2}\mathbb{E}[C_{II,2}^v] \leq \frac{3}{2}(\Delta(H)\mathbb{E}[C_I^v] + \mathbb{E}[C_{II,1}^v] + \mathbb{E}[C_{II,2}^v])$, which is at most

$$\begin{aligned}
\frac{3}{2} \cdot 3 \cdot \frac{q^4\Delta(H)^2}{2p^5} &\leq \frac{9}{4} \cdot \frac{q^4 d(H)^2 \log^{16} n(H)}{p^5} \\
&= \frac{q}{4p}\frac{d(H)}{4}\frac{36q^3 d(H)\log^{16} n(H)}{p^4} \\
&\leq \frac{q}{4p}\delta(H)36d(H)^{3sr+1-4r}\log^{16} n(H).
\end{aligned}$$

Hence, by (A.3),

$$\frac{3}{2} \cdot 3 \cdot \frac{q^4\Delta(H)^2}{2p^5} \leq \frac{q}{4p}\delta(H).$$

Therefore, with probability $1-o(1)$, $\delta(J') \geq \frac{q}{2p}\delta(H) - \frac{9}{4} \cdot \frac{q^4 d(H)^2 \log^{16} n(H)}{p^5} \geq \frac{q}{4p}\delta(H)$. This completes the proof of (iii). ∎

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