1. Analysis of Algorithms

Describe an algorithm for deciding if an n-vertex graph G contains a clique of size 6. Explain how to modify the algorithm so it would also find such a clique in G (if one exists). The running time of both algorithms should be $O(n^5)$.
Hint. You may wish to consider a graph with vertex-set $E(G)$ and suitably defined adjacency.

Solution: Given G let $m = |E|$ and define an m-vertex graph T as follows. Each vertex of T represents an edges of G. We connect two vertices $(u, u'), (v, v')$ of T if and only if u, u', v, v' form a clique of size 4 in G. Then it is easy to see that G contains a clique of size 6 if and only if T contains a triangle. Now we can use fast matrix multiplication to decide in time $O(m^\omega) = O(n^{2\omega}) \ll O(n^5)$ if T contains a triangle. In order to actually find such a triangle, we can use the algorithm we saw in class that finds witnesses for Boolean matrix multiplication in time $O(n^\omega)$.

2. Approximation Algorithms

Let $G = (V, E)$ be a complete graph with distances on its edges; the distance between two vertices u and v is given by $d(u, v)$ and the distances satisfy the triangle inequality. The *k-partition* problem is to partition V into k subsets, $C_1, C_2, \ldots, C_k$, so that the maximum distance between any pair of vertices in the same subset is minimized. Formally, define the *diameter* of a subset $C_r \subset V$ as

$$\text{Diam}(C_r) = \max\{d(v_i, v_j) : v_i, v_j \in C_r\}$$

Then we wish to find a partition that minimizes $\max\limits_{r\in\{1,2,\ldots,k\}} \text{Diam}(C_r)$.

Now consider the following process. Start with an arbitrary vertex. Call it v_1. Then at the i^{th} step, $i \geq 2$, let

$$\delta_i = \max_u \min_{j\in\{1,2,\ldots,i-1\}} d(u, v_j)$$

and define v_i to be the vertex u that achieves the maximum. That is, v_i is the vertex u that maximizes the minimum distance of u to one of the vertices in $\{v_1, v_2, \ldots, v_{i-1}\}$.

1. Show that δ_{k+1} is a lower bound on the value of the optimal solution of the k-partition problem.
2. Give an efficient 2-approximation algorithm for the k-partition problem.

Solution. Note that δ_i is a nondecreasing sequence. Let $v_1, \ldots, v_{k+1}$ be the first $k + 1$ points in the sequence found by the process described. Consider the partition on them induced by the optimal k-partition. At least two of the vertices, say v_i and v_j, with $i < j$, must be in the same part of the optimal partition. This implies that the diameter of the part they lie in must be at least δ_j. Therefore, the optimal partition has at least one part of diameter at least δ_{k+1}, i.e., $\text{OPT} \geq \delta_{k+1}$.

For the second part, find the first k vertices according to the process. Call these the anchors of a k-partition. For every vertex $u \notin \{v_1, \ldots, v_k\}$, assign it part i if v_i is the closest to u among the anchors (break ties arbitrarily). Then for each part i, for any vertex u in the part, $d(u, v_i) \leq \delta_{k+1}$. Therefore, by the triangle inequality, for any two vertices u, v in the same part i,

$$d(u, v) \leq d(u, v_i) + d(v_i, u) \leq 2\delta_{k+1} \leq 2\text{OPT}.$$

3. Theory of Linear Inequalities

Let $d, f \in \mathbf{R}^n$ be integer vectors with all components positive and let t be a positive integer. Suppose $d_i \leq t$ for all $i = 1, \ldots n$, where $d = (d_1, \ldots, d_n)^T$. Let A be a matrix such that columns of A are the non-negative integer solutions to the inequality $d^T x \leq t$. The integer cutting-stock problem is

$$\min(e^T y : Ay = f, y \geq 0, y \text{ integer}) \tag{1}$$

where e is the vector of all 1's. Show that (1) has an optimal solution with at most 2^n positive components.

Solution. A solution is available upon request.

4. Combinatorial Optimization

Let $G = (V, E)$ be a complete graph having an even number of vertices and let $c = (c_e : e \in E)$ be edge weights such that $c \geq 0$ and c satisfies the triangle inequality. For $X \subseteq V$ let $\delta(X)$ denote the set of edges with one end in X and the other end in $V - X$. Let $\mathcal{C}$ denote the set of all sets D of the form $D = \delta(X)$ such that $X \subseteq V$, $|X| \geq 3$, $|V(G) - X| \geq 3$ and $|X|$ is odd. The dual LP for Edmonds' perfect-matching system is

$$\begin{gathered} \text{Maximize} \sum(y_v : \ v \in V) + \sum(Y_D : \ D \in \mathcal{C}) \\ \text{subject to} \\ y_v + y_w + \sum(Y_D : \ e \in D \in \mathcal{C}) \leq c_e, \ \text{for all } e = vw \in E \\ Y_D \geq 0, \ \text{for all } D \in \mathcal{C}. \end{gathered}$$

Show that there exists an optimal dual solution such that $y_v \geq 0$ for all $v \in V$.

Solution. A solution is available upon request.

5. Graph Theory

Let $k \geq 2$ be an integer. Prove that in a k-connected graph, for every set of k vertices there is a cycle that includes all of them.

Solution: For $k = 2$ this follows directly from Menger's theorem. For $k > 2$ there is, by induction, a cycle C containing $k - 1$ of the given vertices, and we may assume that the last vertex, say v, is not on C. The $k - 1$ given vertices on C divide C into $k - 1$ edge-disjoint paths. Let us call those paths *segments*. If $|V(C)| = k - 1$ (that is, $V(C)$ consists entirely of the given vertices), then let $l := k - 1$; otherwise let $l := k$. By Menger's theorem there exist l paths from v to $V(C)$, vertex-disjoint, except for v. It follows that some two of those paths, say P and Q, have ends in the same segment, and hence $C \cup P \cup Q$ contains a cycle that includes all the given vertices.

6. Probabilistic methods

Let $G = (V, E)$ be a graph with n vertices and m edges. Let $t \geq 1$ be arbitrary.

(i) Form a (random) subset T of $V(G)$ by picking a (uniformly) random vertex of the graph t times, with repetition. (Thus $|T| \leq t$.) Let $N(T)$ denote its common neighborhood – the set of vertices adjacent to *every* vertex of T. Let $X = |N(T)|$.

Show that

$$E[X] \geq \frac{(2m)^t}{n^{2t-1}}.$$

(ii) Suppose that

$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t \geq u.$$

Then prove that there exists a subset $U \subset V(G)$ of at least u vertices, such that every set of s vertices in U has at least k common neighbors.

Solution: (i) Note that the probability that a vertex v is in $N(T)$ is just the probability that T is a subset of its neighborhood. Hence, by the convexity of x^t (for $t \geq 1$),

$$E(X) = \sum_{v \in V} \left(\frac{|N(v)|}{n}\right)^t \geq n\left(\frac{1}{n}\sum_{v \in V}\frac{|N(v)|}{n}\right)^t = \frac{(2m)^t}{n^{2t-1}}.$$

(ii) (Use the deletion method.) Let $A := N(T)$. Let Y denote the number of s-sets in A with at most k common neighbors. Suppose the pair $\{u, v\}$ has at most k common neighbors; then the probability that a $\{u, v\} \subset A$ is at most $(k/n)^t$, since each element of T must lie in the common neighborhood of u and v; the same argument holds for subsets of s vertices, rather than pairs. And so

$$E(Y) \leq \binom{n}{s}(k/n)^t.$$

By linearity of expectation,

$$E[X - Y] \geq \frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t \geq u,$$

and thus there must exist a choice of T such that $X - Y \geq u$. (As usual), simply remove one element from each s-set in A with at most k neighbors, to obtain U as required.

7. Algebra

Prove that any finite subgroup of the multiplicative group of a field is cyclic.

Solution: Let $\mathbf{F}$ be a field and G be a finite subgroup of the group $\mathbf{F}^\times = \mathbf{F}\backslash\{0\}$ under multiplication. Since G is finite and abelian, by the Structure Theorem for Abelian Groups, G is a direct product of finitely many cyclic groups, i.e. $G \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ for some integers $n_1, n_2, \ldots, n_k \geq 2$. It suffices to show that $\gcd(n_i, n_j) = 1$ if $i \neq j$. For $i \neq j$, suppose there is a prime p dividing both n_i and n_j. Then it follows from Sylow Theorem that C_{n_i} and C_{n_j} both contain elements of order p. Since p is prime, if a has order p, then so does $a^2, \ldots, a^{p-1}$. Hence both C_{n_i} and C_{n_j} contain at least $p-1$ elements of order p. However, in a field F, the polynomial $x^p - 1$ has at most $p-1$ roots other than 1, so C_{n_i} and C_{n_j} have a non-empty intersection, which cannot happen in a direct product.