1. Analysis of Algorithms

Describe an algorithm for deciding if an n-vertex graph G contains a clique of size 6. Explain how to modify the algorithm so it would also find such a clique in G (if one exists). The running time of both algorithms should be $O(n^5)$.
Hint. You may wish to consider a graph with vertex-set $E(G)$ and suitably defined adjacency.

2. Approximation Algorithms

Let $G = (V, E)$ be a complete graph with distances on its edges; the distance between two vertices u and v is given by $d(u, v)$ and the distances satisfy the triangle inequality. The *k-partition* problem is to partition V into k subsets, $C_1, C_2, \ldots, C_k$, so that the maximum distance between any pair of vertices in the same subset is minimized. Formally, define the *diameter* of a subset $C_r \subset V$ as

$$\text{Diam}(C_r) = \max\{d(v_i, v_j) : v_i, v_j \in C_r\}$$

Then we wish to find a partition that minimizes $\max_{r \in \{1,2,\ldots,k\}} \text{Diam}(C_r)$.

Now consider the following process. Start with an arbitrary vertex. Call it v_1. Then at the i^{th} step, $i \geq 2$, let

$$\delta_i = \max_u \min_{j \in \{1,2,\ldots,i-1\}} d(u, v_j)$$

and define v_i to be the vertex u that achieves the maximum. That is, v_i is the vertex u that maximizes the minimum distance of u to one of the vertices in $\{v_1, v_2, \ldots, v_{i-1}\}$.

1. Show that δ_{k+1} is a lower bound on the value of the optimal solution of the k-partition problem.
2. Give an efficient 2-approximation algorithm for the k-partition problem.

3. Theory of Linear Inequalities

Let $d, f \in \mathbf{R}^n$ be integer vectors with all components positive and let t be a positive integer. Suppose $d_i \leq t$ for all $i = 1, \ldots n$, where $d = (d_1, \ldots, d_n)^T$. Let A be a matrix such that columns of A are the non-negative integer solutions to the inequality $d^T x \leq t$. The integer cutting-stock problem is

$$\min(e^T y : Ay = f, y \geq 0, y \text{ integer}) \tag{1}$$

where e is the vector of all 1's. Show that (1) has an optimal solution with at most 2^n positive components.

4. Combinatorial Optimization

Let $G = (V, E)$ be a complete graph having an even number of vertices and let $c = (c_e : e \in E)$ be edge weights such that $c \geq 0$ and c satisfies the triangle inequality. For $X \subseteq V$ let $\delta(X)$ denote the set of edges with one end in X and the other end in $V - X$. Let $\mathcal{C}$ denote the set of all sets D of the form $D = \delta(X)$ such that $X \subseteq V$, $|X| \geq 3$, $|V(G) - X| \geq 3$ and $|X|$ is odd. The dual LP for Edmonds' perfect-matching system is

$$\text{Maximize} \sum(y_v : \ v \in V) + \sum(Y_D : \ D \in \mathcal{C})$$
$$\text{subject to}$$
$$y_v + y_w + \sum(Y_D : \ e \in D \in \mathcal{C}) \leq c_e, \text{ for all } e = vw \in E$$
$$Y_D \geq 0, \text{ for all } D \in \mathcal{C}.$$

Show that there exists an optimal dual solution such that $y_v \geq 0$ for all $v \in V$.

5. Graph Theory

Let $k \geq 2$ be an integer. Prove that in a k-connected graph, for every set of k vertices there is a cycle that includes all of them.

6. Probabilistic methods

Let $G = (V, E)$ be a graph with n vertices and m edges. Let $t \geq 1$ be arbitrary.

(i) Form a (random) subset T of $V(G)$ by picking a (uniformly) random vertex of the graph t times, with repetition. (Thus $|T| \leq t$.) Let $N(T)$ denote its common neighborhood – the set of vertices adjacent to *every* vertex of T. Let $X = |N(T)|$.

Show that

$$E[X] \geq \frac{(2m)^t}{n^{2t-1}}.$$

(ii) Suppose that

$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t \geq u.$$

Then prove that there exists a subset $U \subset V(G)$ of at least u vertices, such that every set of s vertices in U has at least k common neighbors.

7. Algebra

Prove that any finite subgroup of the multiplicative group of a field is cyclic.