1. Graph Theory

Let G be a planar triangulation on $n \geq 4$ vertices, let G have no loops or parallel edges, and let $d_1, d_2, \ldots, d_n$ be the degree sequence of G. Prove that $d_1^2 + d_2^2 + \cdots + d_n^2 \leq 2n^2 + 12n - 44$, and for every integer $n \geq 4$ construct a planar triangulation on n vertices with no loops or parallel edges for which equality holds. (You may want to first prove the inequality for triangulations of minimum degree at least four, and then proceed by induction.)

2. Probability

1. The interval $[0,1]$ is partitioned into n disjoint sub-intervals with lengths $p_1, p_2, \ldots, p_n$, and the *entropy* of this partition is defined to be

$$h = -\sum_{i=1}^{n} p_i \log p_i.$$

Let $X_1, X_2, \ldots$ be independent random variables having uniform distribution on $[0,1]$, and let $Z_m(i)$ be the number of the $X_1, X_2, \ldots . X_m$ which lie in the i-th interval of the partition above. Define the quantity

$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}.$$

Prove that, almost surely,

$$\lim_{m\to\infty} \frac{\log R_m}{m} = -h.$$

2. Suppose that an airplane engine will fail, when in flight, with probability $1-p$ independently from engine to engine. Suppose that the airplane will make a successful flight if at least 50% of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

3. Analysis of Algorithms

Let $G = (V, E)$ be an undirected graph. A subset $V' \subseteq V$ is a vertex cover for G if every edge has at least one end in V'. The k-vertex cover problem is the following: given an undirected graph G and an integer $k \leq |V|$, find a vertex cover consisting of at most k vertices. The k-vertex cover problem is $\mathcal{NP}$-Hard. Using the following observation, devise an $O(|V|k + k^2 2^k)$ algorithm to find a k-vertex cover in a graph if one exists:

Observation: For each vertex v, either v or all its adjacent vertices have to be in a vertex cover. Therefore, if v is not in a vertex cover C then all its neighbors have to be in C. Let v be a vertex with degree $> k$. Suppose v is not in a k-vertex cover C. By the previous observation, all its neighbors have to be in the vertex cover C. But, they cannot all be in C since there are more than k neighbors of v. Therefore, every vertex with degree $> k$ must be part of a k-vertex cover.

4. Combinatorial Optimization

Consider two systems of linear inequalities $Ax \leq b$ and $Cx \leq d$, and let $P = \{x : Ax \leq b\}$ and $Q = \{x : Cx \leq d\}$. Let x^* be a specified vector. Formulate, as a polynomial-sized (in the size of the two linear systems) linear-programming model, the problem of determining if there exists an inequality $w^T x \leq \delta$ that is valid for P and Q, but violated by x^*. Modify (if necessary) your LP model so that if such an inequality exists, the model finds one that maximizes the violation $w^T x^* - \delta$ among all vectors w with L_1 norm $\|w\|_1 = 1$.

5. Theory of Linear Inequalities

Let $G = (V, E)$ be a graph with nodes V and edges E. Edmonds' matching-polyhedron theorem implies that the linear-programming problem

$$\max \sum (w_e x_e : e \in E) \tag{1}$$

$$\sum (x_e : v \text{ is an end of } e) \leq 1 \quad \forall\, v \in V, \tag{2}$$

$$\sum (x_e : \text{both ends of } e \text{ are in } S) \leq (|S| - 1)/2 \quad \forall\, S \subseteq V, |S| \text{ odd}, \tag{3}$$

$$x_e \geq 0, \ \forall\, e \in E \tag{4}$$

has an integer optimal solution for any objective vector $w = (w_e : e \in E)$. If each component of w is integer, then it is known that there exists an optimal solution to the dual LP of (1) such that the dual variables corresponding to constraints (2) are half-integer valued and the dual variables corresponding to constraints (3) are integer valued. Use this fact to show that the linear system consisting of inequalities (2), (3), (4) is totally dual integral.

NOTE: No credit will be given for just stating that it is already known that the system (2), (3), (4) is totally dual integral.

6. Algebra

Prove that if G is a group having no subgroup of index 2, then any subgroup of index 3 is normal in G.

7. Randomized Algorithms

For an undirected graph $G = (V, E)$, a 3-way cut is a subset S of edges whose removal from G breaks the graph into at least 3 components. The size of the 3-way cut is the number of edges in S. Use a randomized algorithm to prove that there are $O(n^4)$ 3-way cuts of minimum size.

7. Approximation Algorithms

1. Consider TSP (Traveling Salesman Problem) in the special case of directed graphs, called $\{1,2\}$-graphs. These are complete graphs, with each directed edge of weight 1 or 2. (Note: If u and v are vertices of a complete directed graph, then both the directed edges (u, v) and (v, u) are present. If the edges of the graph are weighted, then the weights of (u, v) and (v, u) may be different). (a) Give a polynomial time 3/2 approximation algorithm for TSP on $\{1,2\}$-graphs. Hint: Start by giving a polynomial time algorithm that finds a minimum weight cycle cover (a set of disjoint cycles such that each vertex belongs to exactly one cycle). (b) Can you give a polynomial time algorithm with approximation guarantee 4/3?

2. Consider the following generalization of the standard metric facility location problem. G is a bipartite graph with bipartition (F, C), where F is a set of *facilities* and C is a set of *cities*. Let f_i be the cost of opening facility i and let c_{ij} be the cost of serving one unit of demand for city j by connecting it to (open) facility i. The connection costs satisfy triangular inequality. In addition, each city j has a nonnegative demand d_{ij}, and any open facility can serve this demand. The cost of serving this demand via open facility i is $c_{ij}d_j$. The problem is to find a subset $I \subseteq F$ of facilities that should be opened, and a function $\phi : C \to I$, assigning cities to open facilities in such a way that the total cost of opening facilities and serving the demand of all cities by connecting them to open facilities is minimized. The IP expressing metric facility location with general city demands is:

$$\begin{array}{lll} \text{minimize} & \sum_{i\in F, j\in C} c_{ij} d_j x_{ij} + \sum_{i\in F} f_i y_i & \\ \text{sunject to} & \sum_{i\in F} x_{ij} \geq 1 & j \in C \\ & y_i - x_{ij} \geq 0 & i \in F, j \in C \\ & x_{ij} \in \{0,1\} & i \in F, j \in C \\ & y_i \in \{0,1\} & i \in F \end{array}$$

Give a polynomial time factor 3 approximation algorithm for the above IP. Hint: Write an LP-relaxation, and proceed along the standard primal dual factor 3 approximation algorithm for metric facility location. However, raise the dual variables a_j at rate d_j.

7. Computational Complexity

Let $CLQ = \{G, k \mid G \text{ is a graph}, k \text{ is an integer and } G \text{ has a clique of size } k\}$.
Let $\mathcal{S} = \{C \mid C \text{ is a circuit with } m \text{ inputs that accepts all length } m \text{ encodings of elements of } CLQ\}$.
1. Show that $\mathcal{S}$ is in $\mathcal{CONP}$.
2. Prove: If for all integers $m > 0$, there exists a circuit C_m with m inputs and size $O(m^k)$, for some constant k, such that C_m accepts all length m encodings of elements of CLQ, then $\Pi_2^P \subseteq \Sigma_2^P$.