**ACO Comprehensive Exam** **19 March 2007**

## 1. Graph Theory

Let $G$ be a connected simple graph that is not a cycle and is not complete. Prove that there exist distinct non-adjacent vertices $u, v \in V(G)$ such that the graph obtained from $G$ by deleting both $u$ and $v$ is connected.

## 2. Probability

Let $\{X_n\}$ be a sequence of independent identically distributed random variables. Let

$$S_n := X_1 + \cdots + X_n.$$

Show that

$$\frac{S_n}{\log n} \to 0 \ \text{ a.s.}$$

implies that, for all $c > 0$, $\mathbb{E}e^{c|X_1|} < +\infty$.

## 3. Analysis of Algorithms

**1.** Let $G = (V, E)$ be a graph and let $w : E \to \mathbf{R}^+$ be an assignment of nonnegative weights to its edges. For $u, v \in V$ let $f(u, v)$ denote the weight of a minimum $u$–$v$ cut in $G$. Show that for $u, v, w \in V$,

$$f(u, v) \geq \min\{f(u, w), f(w, v)\}.$$

Generalize this to show that for $u, v, w_1, \ldots, w_r \in V$,

$$f(u, w) \geq \min\{f(u, w_1), f(w_1, w_2), \ldots, f(w_r, v)\}.$$

**2.** Let $T$ be a tree on a vertex set $V$ with weight function $w'$ on its edges. We will say that $T$ is a *flow equivalent tree* if it satisfies the following condition: for each pair of vertices $u, v \in V$, the weight of a minimum $u$–$v$ cut in $G$ is the same as that in $T$. Let $K$ be the complete graph on $V$. Define the weight of each edge $(u, v)$ in $K$ to be $f(u, v)$. Show that any maximum weight spanning tree in $K$ is a flow equivalent tree for $G$.

## 4. Linear Programming

I found a damaged sheet with the data of a linear programming program. This is what was on the sheet:

<u>Problem:</u>

$$\begin{array}{lrrrrcr}
\text{minimize} & 9x_1 & -2x_2 & -12x_3 & +31x_4 & & \\
\text{s.t.} & & & & & & \\
& 1?x_1 & -x_2 & -2x_3 & +2?x_4 & \geq & 9 \\
& -1?x_1 & -x_2 & -1?x_3 & +2x_4 & \geq & 10 \\
& ?x_1 & +??x_2 & -??x_3 & -?x_4 & \geq & ? \\
& -?x_1 & +??x_2 & +??x_3 & -?x_4 & \geq & ? \\
& ??x_1 & +?x_2 & +?x_3 & +?x_4 & \geq & -?? \\
& & & x_1, x_2, x_3, x_4 \geq 0 & & &
\end{array} \tag{1}$$

Solution:

< computations >

Answer: The optimal value is 1?.

Above, "?" stands for a decimal digit 0,1,...,9, perhaps different in different places. What is the optimal value in the problem? Justify your answer.

### 5. Combinatorial Optimization

Given a set of positive numbers $b_1, \ldots, b_n$, consider the following mixed-integer set

$$S = \{(x, y) \in \Re_+ \times \{0,1\}^n : \quad x + ay_i \geq b_i \;\; i = 1, \ldots, n\},$$

where $a \geq \max\{b_i : \; i = 1, \ldots, n\}$. Consider a subset $R := \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ indexed such that $0 =: b_{i_0} < b_{i_1} \leq b_{i_2} \leq \cdots \leq b_{i_r}$, and the corresponding inequality

$$x + \sum_{k=1}^{r} (b_{i_k} - b_{i_{k-1}}) y_{i_k} \geq b_{i_r}. \qquad (1)$$

**1.** Prove that the above inequality (for any subset $R$) is valid for $\text{conv}(S)$.
**2.** Show that the above family of inequalities can be separated in polynomial time by solving an appropriate shortest path problem.

### 6. Algebra

Let $G$ be a group of order 203. If $H$ is a normal subgroup of $G$ of order 7, then show that $H$ is contained in the center of $G$ and that $G$ is abelian.

### 7. Approximation Algorithms

Let $k$ be a power of two. Consider the following generalization of the Steiner forest problem to higher connectivity requirements: the specified connectivity requirement function $r$ maps pairs of vertices to $\{0, \ldots, k\}$, where $k$ is part of the input. Assume that multiple copies of any edge can be used; each copy of edge $e$ will cost $c(e)$. Give a factor $2 \cdot (\log_2 k + 1)$ algorithm for the problem of finding a minimum cost graph satisfying all connectivity requirements. You are allowed to use the Goemans-Williamson factor 2 Steiner forest algorithm as a subroutine.

### 7. Randomized Algorithms

Let $n$ be an odd number. There are $n$ cities $\{C_1, C_2, \ldots, C_n\}$ located at equidistant points on a circle. Two cars move at random from city to city, dropping packages at random in the visited cities. In particular, the process works as follows. At every time:

- Each car independently, chooses uniformly one of the two adjacent cities to the one where it is located and moves to it.

- With probability $p = n^{-4}$, the car in the city with the smallest number of packages dropped (among the two cities where the two cars are located) drops a new package. (Ties are broken randomly.)

1. Upper bound the number of packages in the city with the maximum number of packages. This should be a high probability bound (i.e., with probability tending to 1 when $n$ tends to infinity).
2. What would be the result if the $n$ cities were located at the vertices of a $d$-regular graph (instead of the cycle)? (Each car does a random walk on the vertices of the graph.)