1. Computability, Complexity and Algorithms

Let $G = (V, E)$ be an undirected graph. Consider the following algorithm to find a large matching in G:

1. Start with $M = \emptyset$, the empty matching.
2. Add edges of G greedily to M as long as they maintain a matching.
3. If there is any edge $(u, v) \in M$ such that removing (u, v) from M allows you to add 2 new edges, then apply this change, increasing the size of M by one. Repeat this step as long as such a change is possible (augmentations of length 3).

- Show that the resulting matching M has at least $2/3$ as many edges as a maximum matching of G.
- Consider the extension where the algorithm augments on paths of length up to $2k + 1$. Show that the matching obtained has size at least $(k + 1)/(k + 2)$ times the size of the maximum cardinality matching.
- Suppose G has nonnegative weights on its edges. Show that any greedy maximal matching — choose edges in order of weight while maintaining a matching — gives a matching of at least half the weight of a maximum weight matching.

Solution: Let N be an maximum matching of G. Consider the symmetric difference of M and N. It is a graph with degrees in $\{0, 1, 2\}$, therefore consisting of isolated vertices, paths and cycles. All cycles must be even alternating cycles. On cycles and even paths, both matchings have the same number of edges as the paths must be alternating. There can be no odd paths of length 3, since this would imply either an augmenting path of length 3 (ruled out by the conclusion of the algorithm) or a larger matching than N (ruled out by the optimality of N). On any path of length 5, the matching M has at least $2/3$ as many edges as N. All the cycles must be even.

For the second part, we observe that on paths of length at least $2k + 1$, M has the required fraction of edges compared to N.

For the third part, notice that the weight of any edge of the optimal matching is at most the weight of one of its neigboring edges in the symmetric difference, since if it were heavier than both, it would be picked before its neighbors. Adding up over all edges we see that the optimum matching is at most twice the greedy matching, since each edge of the latter is counted at most twice.

2. Analysis of Algorithms

Given an edge weighted complete bipartite graph $G = (V, E)$ and a perfect matching M in G, define $f(M)$ to be the weight of the heaviest edge in M. Define a *bottleneck perfect matching* in G to be a perfect matching N that minimizes $f(N)$.

First consider the algorithm that simply finds a minimum weight perfect matching in G. Give an example to show that the matching found by this algorithm may not be a bottleneck perfect matching. What is the approximation ratio achieved by this algorithm?

Give a polynomial time algorithm for finding a bottleneck perfect matching. Make sure your algorithm is as efficient as possible. What is its running time?

Solution: The ratio is unbounded. Let k be a fixed positive number. In $K_{n,n}$, assume that each edge (i, i) has weight k, for $1 \leq i \leq n$, edge $(i, i+1)$ has weight 1 for $1 \leq i \leq n-1$ and edge $(n, 1)$ has weight $n(k-1)$. The rest of the edges are very heavy. Let M be the perfect matching (i, i) for $1 \leq i \leq n$ and N be the perfect matching consisting of the edges of weight 1 and $n(k-1)$. Now, M is the bottleneck perfect matching and $f(M) = k$. The minimum weight perfect matching is N and $f(N) = n(k-1)$. As n goes to infinity, the ratio is unbounded.

An algorithm is as follows: Do a binary search on edge weights. While considering weight w, pick only the edges of weight at most w and check if they contain a perfect matching. In this manner, find the minimum w such that the edges of weight at most w have a perfect matching. The running time is $O(n^{2.5} \log n)$.

3. Theory of Linear Inequalities

Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope. Let $\text{vert}(P)$ be the set of vertices of P. Let $X \subseteq \text{vert}(P)$. Define $P(X) := \text{conv}\,(\text{vert}(P) \setminus X)$. The graph of the polytope P is a graph G_P with nodes corresponding to $\text{vert}(P)$ such that two nodes are adjacent in G_P if and only if the corresponding vertices are adjacent in P (i.e. the two vertices are contained in a one-dimensional face of P).

Let $X \subseteq \text{vert}(P)$ and let $(X_1, \ldots, X_m)$ be a partition of X such that X_i and X_j are independent in G_P, i.e. there is no edge connecting X_i to X_j for all $1 \leq i < j \leq m$. Then show that

$$P(X) = \bigcap_{i=1}^{m} P(X_i).$$

Solution. Since $P(X) \subseteq P(X_i)$, we have that $P(X) \subseteq \bigcap_{i=1}^m P(X_i)$. Therefore it is sufficient to prove that $P(X) \supseteq \bigcap_{i=1}^m P(X_i)$.

In order to verify this, we will show that for all $c \in \mathbb{R}^n$, we have that

$$\max\{c^\top x \mid x \in P(X)\} \geq \max\left\{ c^\top x \;\middle|\; x \in \bigcap_{i=1}^{m} P(X_i) \right\}. \tag{1}$$

Let u be an optimal solution to the right-hand-side of (1). Let W be the set of vertices w of P such that $c^\top w \geq c^\top u$. Observe that the set of nodes corresponding to W in G_p is connected. (Indeed, the set of vertices of P that maximize $c^\top x$, are all the vertices of a face of P and thus connected. Running simplex algorithm starting from any non-optimal vertex $w \in W$ shows that w is connected to some optimal vertex that maximizes $c^\top x$.)

Since $(X_1, \ldots, X_m)$ are independent and W is connected, there can be only two possible cases:

1. $W \nsubseteq X_1 \cup \cdots \cup X_m$ which implies that there exists $w \in W$, such that $w \notin X_1 \cup \cdots \cup X_m$: In this case, $w \in P(X)$ and $c^\top w \geq c^\top u$ as desired.

2. $W \subseteq X_1 \cup \cdots \cup X_m$ which implies $W \subseteq X_i$ for some i: This case is not possible, since we have the following contradiction; on one hand we have $u \in P(X_i) \subseteq P(W)$ and on the other hand by definition of W we have $c^\top x < c^\top u$ for all $x \in \text{vert}(P) \setminus W$, i.e. $c^\top x < c^\top u$ for all $x \in P(W)$.

4. Combinatorial Optimization

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Let $c(e)$ for $e \in E$ be the capacity of an edge. Furthermore, let $R = \{((s_1, t_1), d_1), ((s_2, t_2), d_2)\}$ be a set of two commodities, i.e., a quantity d_1 has to be send from source s_1 to sink t_1 and a quantity d_2 has to be send from source s_2 to sink t_2. Let $\delta_E(W)$ be the set of edges with exactly one endpoint in W and let $\delta_R(W)$ be the set of commodities with either its source or its sink in W but not both.

Cut condition: For each $W \subseteq V$, the capacity of $\delta_E(W)$ is not less than the demand of $\delta_R(W)$.

Euler condition:

$$\sum_{e \in \delta(v)} c(e) \equiv 0 \ (mod\ 2) \text{ if } v \neq s_1, t_1, s_2, t_2$$

$$d_1 \ (mod\ 2) \text{ if } v = s_1, t_1$$

$$d_2 \ (mod\ 2) \text{ if } v = s_2, t_2$$

We have the following theorem:

Theorem 1 *If all capacities and demands are integer and both the cut condition and the Euler condition are satisfied, then the undirected 2-commodity flow problem has an integer solution.*

Question 1. Prove the following lemma

Lemma 1 *Every cut in an Eulerian graph (with edge capacities equal to one) has even cardinality.*

Question 2. Use Theorem 1 and Lemma 1 to show the following. Let $G = (V, E)$ be an Eulerian graph and let s_1, t_1, s_2, t_2 be distinct vertices. Then the maximum number k of pairwise edge-disjoint paths $P_1, \ldots, P_k$, where each path P_j connects either s_1 and t_1 or s_2 and t_2, is equal to the minimum cardinality of a cut both separating s_1 and t_1 and separating s_2 and t_2.

Solution. A graph is Eulerian if and only if it has no vertices of odd degree. Consider any cut W. We have that $|\sum_{v\in W}\delta_E(v)| = 2|E(W)| + |\delta_E(W)|$, where $E(W)$ denotes the set of edges with both endpoints in W. This implies that $|\delta_E(W)| = |\sum_{v\in W}\delta_E(v)| - 2|E(W)|$, and, since $|\sum_{v\in W}\delta_E(v)|$ is even (the graph is Eulerian), we get the desired result.

Let k^* be the maximum number of pairwise edge-disjoint paths $P_1, \ldots, P_{k^*}$ in G, where each path P_j connects either s_1 and t_1 or s_2 and t_2.

Let the capacity of each edge be one, i.e., $c(e) = 1$ for all $e \in E$. Then we have that there exist k pairwise edge-disjoint paths $P_1, \ldots, P_k$, where each path P_j connects either s_1 and t_1 or s_2 and t_2, if and only if there exist demands d_1 and d_2 such that $d_1 + d_2 = k$ and an integer solution to the undirected 2-commodity flow problem exists.
Let

$$m_1 = \min\Big\{|\delta(W)| \;\Big|\; W \subseteq V,\ \delta_R(W) = (s_1, t_1)\Big\}$$

and

$$m_2 = \min\Big\{|\delta(W)| \;\Big|\; W \subseteq V,\ \delta_R(W) = (s_2, t_2)\Big\}$$

be the cardinality of a minimum cut separating (s_1, t_1) and (s_2, t_2), respectively.
Let

$$k^* = m = \min\Big\{|\delta(W)| \;\Big|\; W \subseteq V,\ \delta_R(W) = \{(s_1, t_1), (s_2, t_2)\}\Big\}$$

be cardinality of a minimum cut separating both (s_1, t_1) and (s_2, t_2). Lemma 1 implies that

m_1, m_2 and m are even. We will show that $m_1 + m_2 \geq m$, which implies that there exist d_1, d_2 with $d_1 + d_2 = m$, $d_1 \leq m_1, d_2 \leq m_2$, and d_1, d_2 even. Note that d_1, d_2 even implies that the

Euler condition is satisfied (since $\sum_{e\in\delta(v)} c(e) \equiv 0\ \forall v$) and that $d_1 + d_2 = m = k^*$ implies that cut condition is satisfied, which in turn implies that the conditions of Theorem 1 are satisfied and an integer solution exists, which gives the desired result. *Claim*: $m_1 + m_2 \geq m$.

Proof. Let W_1 be such that $\delta_R(W_1) = (s_1, t_1)$ and $|\delta(W_1)| = m_1$. Let W_2 be such that $\delta_R(W_2) = (s_2, t_2)$ and $|\delta(W_2)| = m_2$. Now consider the following four cases:

- *Case 1.* $\delta_R(W_1 \cup W_2) = \{(s_1, t_1), (s_2, t_2)\}$.
  This implies $m \leq |\delta_E(W_1 \cup W_2)| \leq |\delta_E(W_1)| + |\delta_E(W_2)| \leq m_1 + m_2$.
- *Case 2.* $\delta_R(W_1 \setminus W_2) = \{(s_1, t_1), (s_2, t_2)\}$.
  This implies $m \leq |\delta_E(W_1 \setminus W_2)| \leq |\delta_E(W_1) \cup \delta_E(W_2)| \leq m_1 + m_2$.
- *Case 3.* $\delta_R(W_2 \setminus W_1) = \{(s_1, t_1), (s_2, t_2)\}$.
  This implies $m \leq |\delta_E(W_2 \setminus W_1)| \leq |\delta_E(W_1) \cup \delta_E(W_2)| \leq m_1 + m_2$.

- *Case 4.* $\delta_R(W_1 \cap W_2) = \{(s_1, t_1), (s_2, t_2)\}$.
  This implies $m \leq |\delta_E(W_2 \cap W_1)| \leq |\delta_E(W_1) \cup \delta_E(W_2)| \leq m_1 + m_2$. $\square$

5. Graph Theory

Let $k \geq 1$ be an integer and let G be a k-connected k-regular graph on an even number of vertices. Prove that G has a perfect matching.

Solution: We will show that for every $X \subseteq V(G)$ we have $o(G \backslash X) \leq |X|$, where $o(H)$ denotes the number of odd components of the graph H. The conclusion then follows from Tutte's 1-factor theorem. Suppose for a contradiction that $o(G \backslash X) > |X|$ for some set $X \subseteq V(G)$, and let N be the number of edges with one end in X and the other end in $V(G) - X$. Then $N \leq k|X|$, because G is k-regular. On the other hand, we claim that for every component K of $G \backslash X$ there are at least k edges with exactly one end in $V(K)$ (and therefore the other end in X). Indeed, let F be the set of edges with exactly one end in $V(K)$, and assume for a contradiction that $|F| < k$. Let X' be the set of ends of edges in F that belong to X. Then $X' = X$, because G is k-connected, and hence $|X| < k$. The k-connectivity of G implies that K is the only component of $G \backslash X$, and hence $1 \geq o(G \backslash X) > |X|$. Thus $X = \emptyset$, and yet $o(G \backslash X) \geq 1$, contrary to the fact that G has an even number of vertices. This proves our claim that for every component K of $G \backslash X$ there are at least k edges with exactly one end in $V(K)$. By the claim $N \geq kc$, where c is the number of components of $G \backslash X$. Thus

$$o(G \backslash X) \leq c \leq N/k \leq |X|,$$

a contradiction.

6. Probabilistic methods

Let $X_1 \ldots, X_n$ be independent random variables with $X_i \in \{0, 1\}$ and $\mathbf{Prob}[X_i = 1] = p$, for $i = 1, \ldots, n$, where $0 < p < 1$. Set $X := \sum_{i=1}^n X_i$. Prove that for any $t \in [0, 1-p]$, we have

$$\mathbf{Prob}[X \geq (p+t)n] \leq e^{-nh(p,t)},$$

where $h(p,t) = (p+t)\ln \frac{p+t}{p} + (1-p-t)\ln \frac{1-p-t}{1-p}$, and is also referred to as a "relative entropy function".

Solution: Let $\lambda > 0$ be a parameter to be determined later. We have

$$\mathbf{Prob}[X \geq (p+t)n] = \mathbf{Prob}[\lambda X \geq \lambda(p+t)n] = \mathbf{Prob}\left[e^{\lambda X} \geq e^{\lambda(p+t)n}\right].$$

From Markov's inequality, we obtain

$$\mathbf{Prob}\left[e^{\lambda X} \geq e^{\lambda(p+t)n}\right] \leq \frac{\mathbf{E}\left[e^{\lambda X}\right]}{e^{\lambda(p+t)n}}.$$

Now, the independence of the X_i yields

$$\mathbf{E}\left[e^{\lambda X}\right] = \mathbf{E}\left[\prod_{i=1}^{n} e^{\lambda X_i}\right] = \prod_{i=1}^{n} \mathbf{E}\left[e^{\lambda X_i}\right] = (pe^{\lambda} + 1 - p)^n.$$

Thus

$$\mathbf{Prob}[X > (p+t)n] \leq \frac{(pe^{\lambda} + 1 - p)^n}{e^{\lambda(p+t)n}},$$

for every $\lambda > 0$.

The right hand-side is minimized when choosing:

$$e^{\lambda} = \frac{(1-p)(p+t)}{p(1-p-t)}.$$

Plugging this into the above inequality, we obtain:

$$\mathbf{Prob}[X > (p+t)n] \leq \left[\left(\frac{p}{p+t}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{1-p-t}\right]^n = e^{-n\left((p+t)\ln\frac{p+t}{p} + (1-p-t)\ln\frac{1-p-t}{1-p}\right)}$$

7. Algebra

(a) Suppose $K \subset H \subset G$ are groups under the same operation and that K is normal in H and H is normal in G. Does K have to be normal in G?

(b) Let G be a group and H be a subgroup of G with index n. Prove that there is a normal subgroup K of G such that $K \subset H$ and $[G : K] \leq n!$.

Solution:

(a) No. Let G be the dihedral group of order 8 consisting of the symmetries (rotations and reflections) of a square. Let H be the subgroup generated by the 180-degree rotation and one reflection s. Then $|H| = 4$ and $[G : H] = 2$, so H is normal in G. Let K be the group consisting of the identity and s only, which is normal is H since H is abelian. But K is not normal in G since $rsr^{-1} \notin K$ for the 90-degree rotation r.

(b) Consider the group G acting on the set of left cosets of H by left multiplication. This gives a homomorphism φ from G to the symmetric group S_n of order $n!$. Let K be the kernel of φ, which is a normal subgroup of G. In particular, for any $x \in K$ we have $xH = H$, so $x \in H$. This shows that $K \subset H$. Moreover, by the First Isomorphism Theorem, the quotient group G/K is isomorphic to a subgroup of S_n, so we have $[G : K] \leq n!$.

7. Linear Algebra

Let $T \in \text{Hom}(V, V)$, where V is an n-dimensional vector space over a field $\mathbb{F}$. (In other words, T is a linear transformation from V to V.)

(i) Show that if $T^m = 0$, but $T^{m-1} \neq 0$, then there is a vector $v \in V$ such that $\{v, Tv, \ldots, T^{m-1}v\}$ is a linear independent set.

(ii) Show that if $T^m = 0$, then $T^n = 0$.

(iii) Show that if $\ker(T) \cap \text{Im}(T) = \{0\}$, then $\ker(T^2) = \ker(T)$. By giving an example, show that the conclusion is false if the assumption $\ker(T) \cap \text{Im}(T) = \{0\}$ does not hold.

Solution:

(i) Since $T^{m-1} \neq 0$, then there exist $v \in V$ such that $T^{m-1}v \neq 0$. Consider the vectors $v, Tv, \ldots, T^{m-1}v$. Suppose these are not linearly independent, then there exist scalar $a_0, \ldots, a_{m-1} \in \mathbb{F}$, not all 0, such that

$$a_0 v + a_1 T v + \ldots a_{m-1} T^{m-1} v = 0 \; .$$

Multiply this relation by T^{m-1}, to obtain that it must be $a_0 = 0$, so that

$$a_1 T v + \ldots a_{m-1} T^{m-1} v = 0 \; .$$

Multiplying by T^{m-2}, gives $a_1 = 0$. Continuing this way, we reach a contradiction.

(ii) If $m \leq n$, the result is obvious. So, assume $m > n$ and, by contradiction, that $T^n \neq 0$. Then, it must be that there is a first index $k \geq 1$ for which $T^{n+k} = 0$, but $T^{n+k-1} \neq 0$. Reasoning as in part (i), we have that there would be a vector w such that

$$\{w, Tw, \ldots, T^n w, \ldots, T^{n+k-1} w\}$$

is a linearly independent set in V. But since $k \geq 1$, we would have more than n linearly independent elements in V, which is however n-dimensional.

(iii) Surely, if $Tv = 0$, then $T^2 v = 0$, so $\ker(T) \subseteq \ker(T^2)$. To show the reverse implication, suppose that there is a $v \in \ker(T^2)$ but $v \notin \ker(T)$. Then it must be $Tv \neq 0$, hence $Tv \in \text{Im}(T)$, and also $Tv \in \ker(T)$, which is a contradiction.

As far as the counterexample, take $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $Te_1 = 0$, but $Te_2 = e_1$, thus $T^2 = 0$ (since it annihilates a basis), and $\ker(T^2) = \mathbb{R}^2$, but $\ker(T)$ is 1-dimensional.