1. Computability, Complexity and Algorithms

Define the class $\mathcal{SNP}$ to be the class of all languages that are accepted by polynomial time nondeterministic Turing machines that have at most polynomial number of accepting computation paths for any $x \in L$. Define the class $\mathcal{ONP}$ to be the class of all languages that are accepted by polynomial time nondeterministic Turing machines that have an odd number of accepting computation paths for any $x \in L$. Show that $\mathcal{SNP} \subseteq \mathcal{ONP}$.

Solution: Let L be a language in $\mathcal{SNP}$ that is accepted by an $\mathcal{NP}$-machine N. For any string $x \in L$, let $q(|x|)$ be the number of accepting computation paths of N on x, where $q(n)$ is a polynomial function.

On input x, consider the following $\mathcal{NP}$-machine M:

- For $i = 1$ TO $q(|x|)$ DO:
  - Guess i distinct computation paths $P_1, P_2, \cdots, P_i$ and verify that these are accepting computation paths of N on x by simulating N on x guided by the paths.

Clearly, M accepts x iff N accepts x.

Suppose $x \in L$. Then, there are at most $q(|x|)$ accepting computation paths for N on x. For each i in $1 \leq i \leq t$, the machine has $C(q(|x|), i)$ accepting computation paths so that it has a total of $2^{q(|x|)} - 1$ accepting computation paths on x.

(Here, $C(n, m)$ stands for the number of ways of choosing m distinct elements from a set of n elements.)

Suppose $x \notin L$. Then, M has 0 accepting computation paths.

2. Analysis of Algorithms

In the knapsack problem we are given distinct objects $a_1, \dots, a_n$. Each object a_i has positive integer value v_i and positive integer weight w_i, $1 \leq i \leq n$. We are also given a positive integer W, the "knapsack capacity". The problem is to find a subset of objects whose total weight does not exceed W and whose total value is maximized. We assume that $w_i \leq W$ for all $i = 1, 2, \dots, n$. Prove that the following greedy algorithm for the knapsack problem achieves an approximation factor of 1/2. First sort the objects according to decreasing ratio of value to weight. That is, $a_1, \dots, a_n$ are such that $\frac{v_1}{w_1} \geq \dots \frac{v_{k-1}}{w_{k-1}} \geq \frac{v_k}{w_k} \geq \dots \frac{v_n}{w_n}$, and let k be such that $\sum_{i=1}^{k-1} w_i \leq W$ while $\sum_{i=1}^{k} w_i > W$. Next, if $\sum_{i=1}^{k-1} v_i \geq v_k$ then output $\{a_1, \dots, a_{k-1}\}$, while if $\sum_{i=1}^{k-1} v_i < v_k$ then output $\{a_k\}$.

Solution: Write knapsack as an (IP), and take the (LP) relaxation and its dual (DP).

$$
\text{(IP)} \quad \max \sum_{i=1}^{n} v_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \leq W, \quad x_i \in \{0,1\} \quad 1 \leq i \leq n
$$

$$
\text{(LP)} \quad \max \sum_{i=1}^{n} v_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \leq W, \quad 0 \leq x_i \leq 1 \quad 1 \leq i \leq n
$$

$$
\text{(DP)} \quad \min \sum_{i=1}^{n} y_i + zW \quad \text{s.t.} \quad y_i + w_i z \geq v_i \quad 1 \leq i \leq n, \quad y_i \geq 0 \quad 1 \leq i \leq n, \quad z \geq 0
$$

CLAIM 1: The following assignment to the x_i's is primal feasible (check by elementary calculations):

$$x_1 = \ldots = x_{k-1} = 1$$
$$x_k = \frac{W-(w_1+\ldots+w_{k-1})}{w_k}$$
$$x_{k+1} = \ldots = x_n = 0$$

CLAIM 2: The following assignment to the y_i's is dual feasible (check by elementary calculations):

$$\begin{array}{ll} y_i = v_i - w_i \frac{v_k}{w_k} & 1 \le i \le k \\ y_i = 0 & k+1 \le i \le n \\ z = \frac{v_k}{w_k} & \end{array}$$

CLAIM 3: For the primal and dual feasible solutions of CLAIMS 1 and 2, the objective values of the (LP) and (DP) are equal. Thus, these solutions are optimal.
PROOF: Verify that

$$\sum_{i=1}^{k-1} v_i + \frac{W-(w_1+\ldots+w_{k-1})}{w_k} v_k = \sum_{i=1}^{k} \left(v_i - w_i \frac{v_k}{w_k} \right) + \frac{v_k}{w_k} W$$

We are now ready to establish the approximation factor:

$$\begin{aligned} \left(\sum_{i=1}^{k-1} v_i \right) + v_k &\ge \sum_{i=1}^{k-1} v_i + \frac{W-(w_1+\ldots+w_{k-1})}{w_k} v_k \\ &= \text{OPT(LP)} \\ &\ge \text{OPT(IP)} \end{aligned}$$

Thus

$$\left(\sum_{i=1}^{k-1} v_i \right) + v_k \ge \text{OPT(IP)}$$

Thus at least one of $\left(\sum_{i=1}^{k-1} v_i \right)$ and v_k is $\ge$ OPT(IP)/2, and the algorithm indeed picks the largest of the $\left(\sum_{i=1}^{k-1} v_i \right)$ and v_k.

3. Theory of Linear Inequalities

Let $P \subseteq \mathbb{R}^n$ be a nonempty polytope. Let x^0 be a vertex of P. Let $x^1, \ldots, x^k$ be all the neighboring vertices of x^0, i.e., all the one dimensional faces of P containing x^0 are of the form $\text{conv}\{x^0, x^t\}$ for $t \in \{1, \ldots, k\}$. Prove that if $x \in P$, then there exists $\lambda_t \ge 0$ for $t \in \{1, \ldots, k\}$ such that

$$x = \sum_{t=1}^{k} \lambda_t (x^t - x^0) + x^0.$$

Solution. Since x^0 is a vertex, i.e. a face of P, there exists a vector $c \in \mathbb{R}^n$ such that

$$cx^0 < cx \;\forall x \in P \setminus \{x^0\}. \tag{1}$$

Let $x^0, \ldots, x^k, x^{k+1}, \ldots, x^r$ be the vertices of P. Since there are a finite number of vertices, by (1), there exists $d \in \mathbb{R}$ such that $cx^0 < d$ and $cx^t > d$ for all $t \in \{1, \ldots, r\}$. Let $Q \subseteq \mathbb{R}^n$ be the polytope $Q := P \cap \{x \in \mathbb{R}^n \,|\, cx = d\}$. Let $v^t = \text{conv}\{x^0, x^t\} \cap \{x \in \mathbb{R}^n \,|\, cx = d\}$ for $t \in \{1, \ldots, k\}$. Since $cx^0 < d < cx^t$, we obtain that v^t is a point.

We claim that the set of points v^t's are exactly the set of vertices of Q: Let u be a vertex of Q. Therefore there are n linearly independent constraints (i.e. constraints whose left-hand-side vectors are linearly independent) of Q that are satisfied at equality by u (This is equation (23), page 104 in textbook). By definition of Q, $cu = d$ and therefore there are atleast $n-1$ linearly independent constraints of P that are satisfied at equality by u. Therefore u belongs to some one dimensional face of P. Since $cu = d$, and $cx^t > d$ for all $t \in \{1, \ldots, r\}$, this one dimensional face is of the form $\text{conv}\{x^0, x^t\}$ for $t \in \{1, \ldots, r\}$. Since all the one dimensional faces of P containing x^0 are $\text{conv}\{x^0, x^t\}$ for $t \in \{1, \ldots, k\}$, we have that $u = \text{conv}\{x^0, x^t\} \cap \{x \in \mathbb{R}^n \,|\, cx = d\}$ for some $t \in \{1, \ldots, k\}$. Conversely, observe that the point v^t satisifes at equality n linearly independent constraints satisfying Q, since there are $n-1$ linearly independent constraints satisfied at equality by the edge $\text{conv}\{x^0, x^t\}$ and the constraint $cx = d$ is the n^{th} linearly independent constraint (since $cx^t \neq cx^0$, $cx = d$ is linearly independent from the other $n-1$ constraints.). Therefore v^t is a vertex of Q.

Representation Theorem (Thm 8.5) applied to Q and the above claim implies that

$$Q = \text{conv}\{\cup_{t=1}^k v^t\} = \text{conv}\{\cup_{t=1}^k \left(\gamma_t(x^t - x^0) + x^0\right)\}, \tag{2}$$

where $\gamma_t \in [0, 1]$ for all $t \in \{1, \ldots, k\}$.

By applying Representation Theorem to P, it is sufficient to prove the statement of the problem for the vertices x^t, $t \in \{k+1, \ldots, r\}$. By construction of c, there exists, $\tilde{x}$ satisfying

$$\tilde{x} \in \text{conv}\{x^0, x^t\}, \qquad c\tilde{x} = d. \tag{3}$$

Therefore, $\tilde{x} \in P \cap \{x \in \mathbb{R}^n \,|\, cx = d\} = Q$. By (2) and (3), we have that $x^t = \mu(\tilde{x} - x^0) + x^0 = \mu\left(\sum_{i=1}^k \tau_i \left(\gamma_i(x^i - x^0) + x^0\right) - x^0\right) + x^0$ for some $\mu > 1$ and $\tau_i \geq 0$ for all $i \in \{1, \ldots, k\}$, $\sum_{i=1}^k \tau_i = 1$, or equivalently $x^t = \sum_{i=1}^k \lambda_i(x^i - x^0) + x^0$ for some $\lambda_i \geq 0$ for all $i \in \{1, \ldots, k\}$. $\square$

4. Combinatorial Optimization

(a) (3 points) Let A be a matrix with entries equal to 0, 1, or -1 of the following form:

$$\begin{bmatrix} \pm 1 & & & & \pm 1 \\ \pm 1 & \pm 1 & & & \\ & \pm 1 & \ddots & & \\ & & \ddots & \pm 1 & \\ & & & \pm 1 & \pm 1 \end{bmatrix}$$

Show that A is totally unimodular if and only if the sum of the entries is equal to $0 (mod\ 4)$.
Let A and B be two totally unimodular $n \times m$ matrices. Assume that $A[i,j] \neq 0$ if and only if $B[i,j] \neq 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Let G be the bipartite graph with vertices $v_1, \dots, v_n, u_1, \dots, u_m$ such that v_i is adjacent u_j if and only if $A[i,j] \neq 0$.
(b) (2 points) Let T be a forest in G. Show that there exists A' which is obtained from A by repeatedly scaling rows and columns by factors of 1 or -1 such that

$$A'[i,j] = B[i,j] \text{ for all } i,j \text{ such that } v_i u_j \in E(T)$$

(c) (5 points) Show that A may be obtained from B by repeatedly scaling rows and columns by factors of 1 or -1.

Solution. **(a)** Observe that there exists A' of the form

$$\begin{bmatrix} 1 & & & & \alpha \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{bmatrix}$$

for some $\alpha \in \{-1, 1\}$ which is obtained from A by resigning rows and columns by -1. By construction, A is totally unimodular if and only if A' is totally unimodular. Note as well that the sum of the entries modulo 4 is the same for A and A', as the sum modulo 4 is unchanged by resigning either a row or column by -1.

Assume that A is an $n \times n$ matrix. By expanding the determinant on the final column, we see that $det(A') \in \{1, -1, 0\}$ if $\alpha = -1$ and n is odd, or alternatively, if $\alpha = 1$ and n is even. Thus, if A is totally unimodular, then the sum of the entries is equal to zero modulo 4.

To see the other direction, we may assume that $\alpha = -1$ if n is odd and $\alpha = 1$ if n is even. Assume A' is not totally modular and pick a $k \times k$ submatrix A'' of A' such that $\det(A'') \notin \{1, 0, -1\}$. Moreover, do so to minimize k. Every row and every column must have at least two non-zero entries; otherwise, we could expand the determinant on a row or column with at most one non-zero entry and by the minimality of k, derive a contradiction to $\det(A'') \notin \{1, 0, -1\}$. But now it follows that $A'' = A'$ and $\det(A') \in \{1, 0, -1\}$, a contradiction.
(b) Assume the claim is false. Pick totally unimodular matrices A and B with auxiliary graph G defined as above, and forest T in G forming a counterexample to the claim. Moreover, assume we pick the counterexample to minimize $|V(G)|$.

Let $v \in V(T)$ be a leaf. Let $\bar{A}$ (respectively $\bar{B}$) be the matrix obtained from A (resp. B) by deleting the row or column of A (resp. B) corresponding to the vertex v. Let $\bar{G}$ be the auxiliary graph corresponding to A. The graph $T - v$ is a forest in $\bar{G}$, and so by our choice of counterexample, there exists a matrix $\bar{A}'$ obtained from $\bar{A}$ by scaling rows and columns of $\bar{A}$ by -1 such that $\bar{A}'[i,j] = \bar{B}[i,j]$ for all i,j such that $v_i u_j \in E(T - v)$. By scaling the same rows and columns of A, we find A' such that $A'[i,j] = B[i,j]$ for all edges $v_i u_j \in E(T - v)$. We can then rescale the row or column corresponding to v to ensure that A' and B agree on the entry corresponding to the unique edge of T incident the vertex v, proving the claim.
(c) Fix a forest T in G containing a maximum number of edges. Let A' be obtained from A by resigning rows and columns by -1 such that

1. $A'[i,j] = B[i,j]$ for all i,j such that $v_iu_j \in E(T)$, and
2. subject to 1, the number of pairs of indices i,j such that $A'[i,j] = B[i,j]$ is maximized.

We may assume that there exists indices i and j such that $A'[i,j] \neq B[i,j]$, as otherwise the theorem is proven. By our choice of T and choice of counterexample to satisfy 1, the edge v_iu_j is contained in a cycle C of G such that for all edges $v_{i'}u_{j'} \in E(C)$ with $\{i',j'\} \neq \{i,j\}$, we have that $A'[i',j'] = B[i',j']$. Pick such a pair of indices i,j and cycle C to minimize $|E(C)|$. It follows that C is an induced cycle in G.

If we now let $\bar{A}'$ be the submatrix of A' given by the rows and columns corresponding to $V(C)$. Similarly, define $\bar{B}$ to be the submatrix of B given by the rows and columns corresponding to $V(C)$. After possibly reordering the columns and rows, we see that both $\bar{A}'$ and $\bar{B}$ are of the form

$$\begin{bmatrix} \pm 1 & & & & \gamma \\ \pm 1 & \pm 1 & & & \\ & \pm 1 & \ddots & & \\ & & \ddots & \pm 1 & \\ & & & \pm 1 & \pm 1 \end{bmatrix}$$

given the fact that C is an induced cycle. Moreover, $\bar{A}'$ and $\bar{B}$ agree at every entry except one, indicated as γ above, corresponding to the edge v_iu_j of G. However, by part a. of the problem, there is only one choice of the value $\gamma \in \{-1,1\}$ which makes the determinant equal to 1 or -1. This contradicts our choice of i and j, proving the claim.

5. Graph Theory

A graph G is *minimally 2-connected* if it is 2-connected and for every edge $e \in E(G)$ the graph $G \backslash e$ is not 2-connected. Prove that every minimally 2-connected graph has a vertex of degree two.

Solution: This follows easily from the ear decomposition theorem. Another proof can be obtained as follows. Let $e \in E(G)$. Since $G \backslash e$ is not 2-connected, it has at least two blocks. By the block structure theorem $G \backslash e$ has an end-block H; that is, a block containing exactly one cutvertex. Let c be the unique cutvertex of H. Let us choose e and H so that H is minimal with respect to taking subgraphs. Since G is 2-connected, one end of e, say v, belongs to $V(H) - \{c\}$. We may assume that v has degree at least three in G, for otherwise we are done. Thus H has at least three vertices, and it follows that it has an edge f not incident with c. Let H' be an end-block of $G \backslash f$ not containing e. Since H' includes an end of f it follows that H' is a subgraph of H; but it does not include f, and hence it is a proper subgraph of H, a contradiction.

6. Probabilistic methods

A random poset of height 2 is formed as follows: The set of minimal elements is $A = \{a_1, a_2, \ldots, a_n\}$, and the set of maximal elements is $B = \{b_1, b_2, \ldots, b_n\}$. For each pair $(a,b) \in A \times B$, $\Pr[a < b] = p$ where $0 \leq p \leq 1$. In general p is a function of n, but here we fix $p = e^{-12}$. Events corresponding to distinct pairs in $A \times B$ are mutually independent. The notation $a \| b$ indicates that an element $a \in A$ is incomparable with an element $b \in B$. For a poset P in this space, let $f(P)$ denote the least positive

integer so that there exist t linear extensions $L_1, L_2, \ldots, L_t$ of P so that for each pair $(a, b) \in A \times B$ with $a \| b$, there is some L_i for which $a > b$ in L_i.
(a) Show that there exists a constant c so that a.s. $f(P) \leq n - cn/\ln n$. *Hint.* Consider linear extensions in which only the bottom two elements of B are specified. The elements of A are inserted into three gaps.
(b) For each $x \in A \cup B$, let $d(x)$ denote the degree of x in P, i.e., the number of elements comparable with x in P. Also, let $\Delta(P)$ denote the maximum value of $d(x)$ taken over all $x \in A \cup B$. Use a second moment method to show that a.s. $\Delta(P) < (1 + o(1))pn$.

Solution: (a) Let c be a constant (we specify the actual size of c later). Set $t = n - cn/\ln n$ and $m = n - t = cn/\ln n$. Form a family L_1, $L_2, \ldots, L_t$ of linear extensions of P as follows. First, choose an arbitrary t-element subset $S = \{s_1, s_2, \ldots, s_t\}$ of B. Label the remaining elements of B as $S' = \{s'_1, s'_2, \ldots, s'_m\}$. In the linear extension L_i, s_i will be the lowest element of B. Choose an arbitrary partition of the integers in $\{1, 2, \ldots, t\}$ into m blocks each of size t/m, and label these blocks as $W_1, W_2, \ldots, W_m$. When the integer i belongs to the block W_j, the element s'_j will be the second lowest element of B in L_i.

For each $i = 1, 2, \ldots, t$, when $i \in W_j$, we insert the elements of A in L_i by placing them into one of three gaps: the bottom gap is immediately under s_i; the middle gap is between s_i and s'_j; and the top gap is immediately above s'_j (and below all other elements of B). The order of elements of A placed into the same gap is arbitrary. In the bottom gap, we place those elements $a \in A$ with $a < s_i$ in P. In the middle gap, we place those elements $a \in A$ with $a \| s_i$ and $a < s'_j$ in P. In the top gap, we place those elements $a \in A$ with $a \| s_i$ and $a \| s'_j$ in P.

Evidently, each S_i is a linear extension. So we need only show that if $a \in A$, $b \in B$ and $a \| b$, there is some L_i with $a > b$ in L_i. This is obviously true if $b = s_i$ for some i. So we assume that $b = s'_j$ for some j. In this case, $a > b$ in some L_i with $i \in W_j$ unless $a < s_i$ for every $i \in W_j$. Now there are nm pairs (a, s_j) and the probability that any such pair is "bad" is $p^{t/m}$. So the expected number of bad pairs is at most $nmp^{t/m}$. Now $t/m \sim \ln n/c$ and $nm < n^2$, so the expected number of bad pairs is $o(1)$ provided $n^2 e^{\ln n \ln p/c} = o(1)$. Since $\ln p = -12$, it suffices to have $4c = -\ln p = 12$, i.e., $c = 3$.
(b) First, focus on the quantities $d(a)$ where $a \in A$. For each $a \in A$, the quantity $d(a)$ is a bernoulli r.v. with mean pn and variance $pn(1-p)$. It follows that $\Pr[d(a) - pn \geq \lambda\sigma] \leq e^{-\lambda^2/2}$. Now $\sigma = \sqrt{pn(1-p)} > n^{1/3}$, being terribly generous. Setting $\lambda = n^{1/3}$, we see that the probability that some $a \in A$ is "bad" because it has degree more than $pn + n^{2/3}$ is less than $ne^{-n^{2/3}/2}$ which certainly goes to zero, i.e., a.s there are no bad elements of A. Dually, a.s. there are no bad elements of B, which implies that a.s. $\Delta(P) \leq pn + n^{2/3} = (1 + o(1))pn$.

7. Algebra

Let F be a field. Assume that $f_1, \ldots, f_k \in F[x]$ are distinct monic irreducible polynomials and $e_1, \ldots, e_k$ are positive integers. Let $I \subset F[x]$ be the ideal generated by $\prod_{i=1}^k f_i^{e_i}$ and let R be the quotient ring $F[x]/I$. How many ideals does R have? How many of them are maximal ideals?

Solution: We'll show that R has $\prod_{i=1}^k (e_i + 1)$ ideals and that k of them are maximal.

Let $\phi : F[x] \to R$ be the quotient homomorphism. The map $\overline{J} \mapsto J = \phi^{-1}(\overline{J})$ gives a bijection between ideals of R and ideals of $F[x]$ which contain I. Moreover, an ideal $\overline{J}$ of R is maximal if and only if $\phi^{-1}(\overline{J})$ is a maximal ideal of $F[x]$. Thus we will count ideals and maximal ideals of $F[x]$ containing I.

Let J be an ideal of $F[x]$ containing I. Since $F[x]$ is a PID, J is generated by a non-zero element $f \in F[x]$ which we may assume to be monic. The containment $I \subset J$ implies that f divides $\prod_{i=1}^{k} f_i^{e_i}$ and distinct monic divisors of $\prod_{i=1}^{k} f_i^{e_i}$ give distinct ideals J containing I. Thus the number of ideals J containing I is the same as the number of monic divisors of $\prod_{i=1}^{k} f_i^{e_i}$. Since the f_i are assumed to be monic, distinct, and irreducible, there are $\prod_{i=1}^{k}(e_i+1)$ such divisors, namely

$$\prod_{i=1}^{k} f_i^{d_i}$$

where $0 \le d_i \le e_i$ for all i.

Since $F[x]$ is a PID, an ideal J is maximal if and only if it is prime, if and only if it is generated by an irreducible polynomial. The monic irreducible divisors of $\prod_{i=1}^{k} f_i^{e_i}$ are precisely the products $\prod_{i=1}^{k} f_i^{d_i}$ where one of the exponents d_i is 1 and the others are 0. Thus there are k maximal ideals of $F[x]$ containing I.