1. Computability, Complexity and Algorithms

Define the class $\mathcal{SNP}$ to be the class of all languages that are accepted by polynomial time nondeterministic Turing machines that have at most polynomial number of accepting computation paths for any $x \in L$. Define the class $\mathcal{ONP}$ to be the class of all languages that are accepted by polynomial time nondeterministic Turing machines that have an odd number of accepting computation paths for any $x \in L$. Show that $\mathcal{SNP} \subseteq \mathcal{ONP}$.

2. Analysis of Algorithms

In the knapsack problem we are given distinct objects $a_1, \ldots, a_n$. Each object a_i has positive integer value v_i and positive integer weight w_i, $1 \leq i \leq n$. We are also given a positive integer W, the "knapsack capacity". The problem is to find a subset of objects whose total weight does not exceed W and whose total value is maximized. We assume that $w_i \leq W$ for all $i = 1, 2, \ldots, n$. Prove that the following greedy algorithm for the knapsack problem achieves an approximation factor of 1/2. First sort the objects according to decreasing ratio of value to weight. That is, $a_1, \ldots, a_n$ are such that $\frac{v_1}{w_1} \geq \ldots \frac{v_{k-1}}{w_{k-1}} \geq \frac{v_k}{w_k} \geq \ldots \frac{v_n}{w_n}$, and let k be such that $\sum_{i=1}^{k-1} w_i \leq W$ while $\sum_{i=1}^{k} w_i > W$. Next, if $\sum_{i=1}^{k-1} v_i \geq v_k$ then output $\{a_1, \ldots, a_{k-1}\}$, while if $\sum_{i=1}^{k-1} v_i < v_k$ then output $\{a_k\}$.

3. Theory of Linear Inequalities

Let $P \subseteq \mathbb{R}^n$ be a nonempty polytope. Let x^0 be a vertex of P. Let $x^1, \ldots, x^k$ be all the neighboring vertices of x^0, i.e., all the one dimensional faces of P containing x^0 are of the form $\text{conv}\{x^0, x^t\}$ for $t \in \{1, \ldots, k\}$. Prove that if $x \in P$, then there exists $\lambda_t \geq 0$ for $t \in \{1, \ldots, k\}$ such that

$$x = \sum_{t=1}^{k} \lambda_t (x^t - x^0) + x^0.$$

4. Combinatorial Optimization

(a) (3 points) Let A be a matrix with entries equal to 0, 1, or -1 of the following form:

$$\begin{bmatrix} \pm1 & & & & \pm1 \\ \pm1 & \pm1 & & & \\ & \pm1 & \ddots & & \\ & & \ddots & \pm1 & \\ & & & \pm1 & \pm1 \end{bmatrix}$$

Show that A is totally unimodular if and only if the sum of the entries is equal to $0 (mod\ 4)$.
Let A and B be two totally unimodular $n \times m$ matrices. Assume that $A[i,j] \neq 0$ if and only if $B[i,j] \neq 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Let G be the bipartite graph with vertices $v_1, \ldots, v_n, u_1, \ldots, u_m$ such that v_i is adjacent u_j if and only if $A[i,j] \neq 0$.

(b) (2 points) Let T be a forest in G. Show that there exists A' which is obtained from A by repeatedly scaling rows and columns by factors of 1 or -1 such that

$$A'[i,j] = B[i,j] \text{ for all } i,j \text{ such that } v_i u_j \in E(T)$$

(c) (5 points) Show that A may be obtained from B by repeatedly scaling rows and columns by factors of 1 or -1.

5. Graph Theory

A graph G is *minimally 2-connected* if it is 2-connected and for every edge $e \in E(G)$ the graph $G \backslash e$ is not 2-connected. Prove that every minimally 2-connected graph has a vertex of degree two.

6. Probabilistic methods

A random poset of height 2 is formed as follows: The set of minimal elements is $A = \{a_1, a_2, \ldots, a_n\}$, and the set of maximal elements is $B = \{b_1, b_2, \ldots, b_n\}$. For each pair $(a,b) \in A \times B$, $\Pr[a < b] = p$ where $0 \le p \le 1$. In general p is a function of n, but here we fix $p = e^{-12}$. Events corresponding to distinct pairs in $A \times B$ are mutually independent. The notation $a \| b$ indicates that an element $a \in A$ is incomparable with an element $b \in B$. For a poset P in this space, let $f(P)$ denote the least positive integer so that there exist t linear extensions $L_1, L_2, \ldots, L_t$ of P so that for each pair $(a,b) \in A \times B$ with $a \| b$, there is some L_i for which $a > b$ in L_i.
(a) Show that there exists a constant c so that a.s. $f(P) \le n - cn/\ln n$. *Hint.* Consider linear extensions in which only the bottom two elements of B are specified. The elements of A are inserted into three gaps.
(b) For each $x \in A \cup B$, let $d(x)$ denote the degree of x in P, i.e., the number of elements comparable with x in P. Also, let $\Delta(P)$ denote the maximum value of $d(x)$ taken over all $x \in A \cup B$. Use a second moment method to show that a.s. $\Delta(P) < (1 + o(1))pn$.

7. Algebra

Let F be a field. Assume that $f_1, \ldots, f_k \in F[x]$ are distinct monic irreducible polynomials and $e_1, \ldots, e_k$ are positive integers. Let $I \subset F[x]$ be the ideal generated by $\prod_{i=1}^{k} f_i^{e_i}$ and let R be the quotient ring $F[x]/I$. How many ideals does R have? How many of them are maximal ideals?