1. Analysis of Algorithms

A k-uniform hypergraph $H = (V, E)$ is composed of a vertex set V and a collection E of subsets of V of size k (so a 2-uniform hypergraph is just a graph). A vertex cover of $H = (V, E)$ is a collection of vertices that intersects all the edges of H.
1. Give a polynomial time k-approximation algorithm for vertex-cover in k-uniform hypergraphs.
2. Give a fixed-parameter algorithm for vertex-cover in k-uniform hypergraph. That is, show that for any k and d there is an algorithm that decides in time $f(k,d) \cdot n^k$ if a k-uniform hypergraph on n vertices has a vertex cover of size d. Here, $f(k,d)$ can be any function of k and d that is *independent* of n.

2. Approximation Algorithms

Recall that MAX-SAT is the following problem: Given a conjunctive normal form formula f on Boolean variables $x_1, \ldots, x_n$, and non-negative weights, w_c, for each clause c of f, find a truth assignment to the Boolean variables that maximizes the total weight of satisfied clauses.
(a) Show that the following is a factor 1/2 approximation algorithm for MAX-SAT. Let τ be an arbitrary truth assignment, and τ' be its complement, i.e., a variable is True in τ if and only if it is False in τ'. Compute the weight of clauses satisfied by τ and τ', then output the better assignment.
(b) Give a tight example: Class of input instances where this algorithm performs as badly as 1/2.

3. Theory of Linear Inequalities

Let $a_1, \ldots, a_k$ be rational vectors. Show that if $\{a_1, \ldots, a_k\}$ is a Hilbert basis then $\{a_1, \ldots, a_k, -a_1\}$ is also a Hilbert basis.

Use this result to give an alternative proof of Theorem 22.2 in A. Shrijver's *Theory of Linear and Integer Programming*: If $Ax \le b, \alpha^T x \le \beta$ is a totally-dual-integral system, then the system $Ax \le b, \alpha^T x = \beta$ is also totally-dual integral.

4. Combinatorial Optimization

Let $D = (V, A)$ be a directed graph with arc costs $(c_a : a \in A)$ and let $r, s \in V$. Show that the problem of finding a minimum-cost simple directed (r,s)-dipath in D containing every vertex in V can be reduced to the problem of finding a maximum-weight common independent set of three matroids.

5. Graph Theory

Let G be a connected graph on n vertices and m edges. For $v \in V(G)$ let $\delta(v)$ denote the set of edges incident with v, and let X be the subspace of $\mathbb{R}^{E(G)}$ consisting of all vectors $\mathbf{y}$ satisfying $\sum_{e\in\delta(v)} y_e = 0$ for every $v \in V(G)$. Determine the dimension of X and prove that your answer is correct.
Hint. The answer depends on whether G is bipartite or not.

6. Probability/Probabilistic methods

Choose **exactly** one of the problems below.

1. Let $X_i, X_2, \ldots,$ be bounded, independent, identically distributed random variables with mean zero. Let $S_n = \sum_{i=1}^{n} X_i$. Show that if $\alpha > 0$ then, *almost surely*,

$$\frac{S_n}{n^{(1/2)+\alpha}} \to 0\,, \qquad \text{as } n \to \infty\,.$$

Hint: First you may want to show that $E(S_n^{2k}) \le C_k n^k$ for $k \ge 1$, and suitable constant C_k.

2. An (n, k, l)-cover is a family $\mathcal{F}$ of k-subsets of an n-element set such that every l-subset is contained in at least one of $A \in \mathcal{F}$. Let $M(n, k, l)$ denote the minimal cardinality of such a cover.

Show that

$$M(n,k,l) \le \frac{\binom{n}{l}}{\binom{k}{l}}\left[1 + \ln\binom{k}{l}\right].$$

7. Algebra

Let V be a finite-dimensional vector space over the complex numbers. Let S and T be linear maps $V \to V$. Assume that S and T commute and that the characteristic polynomial of S has distinct roots. Show that every eigenvector for S is an eigenvector for T. Show that if T is nilpotent (that is, $T^n = 0$ for some $n > 0$), then $T = 0$.