## 1. Graph Theory

Let $G$ be a simple graph with maximum degree $d$. Prove that $E(G)$ can be decomposed into pairwise disjoint (possibly empty) matchings $M_1, \ldots, M_{d+1}$ such that $-1 \leq |M_i| - |M_j| \leq 1$ for all $1 \leq i, j \leq d+1$.

## 2. Probability

Assume that a Markov text $X_0, X_1, X_2, X_3, \ldots, X_n$ with letters from the alphabet $S = \{a, b, c\}$ has its transition probabilities given in the following transition matrix:

$$\begin{pmatrix} p_{a\to a} & p_{a\to b} & p_{a\to c} \\ p_{b\to a} & p_{b\to b} & p_{b\to c} \\ p_{c\to a} & p_{c\to b} & p_{c\to c} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

a) What are the stationary probabilities $\pi(a), \pi(b), \pi(c)$ equal to? (i.e. what are the long term frequencies of the $a$'s, $b$'s and $c$'s in the text?)
b) Given that we start in state $a$, what is the expected time until we visit state $b$ for the first time?

## 3. Analysis of Algorithms

Given an undirected graph $G = (V, E)$, let $C$ be a coloring of $G$ where each vertex is colored either red, green, or blue. We say that an edge is monochromatic in $C$ if the colors on its endpoints agree, and bichromatic if they disagree. An *ideal coloring* is any coloring that maximizes the number of bichromatic edges. Unfortunately finding an ideal coloring is NP-hard.
**(a)** Let $M(G)$ be the number of monochromatic edges in an ideal coloring. Show that it is NP-hard to approximate $M(G)$ to within a factor of $10^{100}$.
**(b)** Let $B(G)$ be the number of bichromatic edges in an ideal coloring. Give a randomized algorithm that outputs a coloring such that the expected number of bichromatic edges is at least $\frac{2}{3}B(G)$.

## 4. Combinatorial Optimization

Let $A$ be an integral $m \times n$ matrix and let $b$ and $c$ be integral $m$-dimensional vectors. Show that there exists an integral vector $x$ with $Ax \in \{b, c\}$ if and only if there does not exist a vector $y$ such that $y^T A$ is integral, $y^T b$ is not an integer, and $y^T c$ is not an integer.

## 5. Theory of Linear Inequalities

Let $P = \{x : Ax \leq b,\ \mathbf{0} \leq x \leq \mathbf{1}\}$ be a rational polytope of dimension $d$, where $A$ is an $m \times n$ matrix, $b$ is an $m$-dimensional vector, and $\mathbf{0}$ and $\mathbf{1}$ represent $n$-dimensional vectors with all components 0 and 1, respectively. Suppose $P$ does not contain any integer vectors. Show that the Chvátal rank of $P$ is no greater than $d$.

## 6. Algebra

Let $G$ be a finite group acting on a set $X$, and let $H$ be a normal subgroup of $G$. If $x \in X$, show that the $G$-orbit of $x$ is a union of at most $[G : H]$ $H$-orbits of $X$, each having the same cardinality.

## 7. Randomized Algorithms

Consider the following scheme for shuffling a deck of $n$ cards labelled $c_1, c_2, \ldots, c_n$. For $i = 1, \ldots, n$, let $X_t(i)$ denote the card in the $i$-th position at time $t$. Let $X_0$ be an arbitrary ordering of the cards. For $t \geq 1$, given $X_{t-1}$ define $X_t$ as follows:

- Choose position $i$ uniformly at random from $\{1, \ldots, n\}$ and card $c_j$ uniformly at random from $\{c_1, \ldots, c_n\}$.
- Swap the card in position $i$ with card $c_j$. I.e., let $X_{t+1}(i) = c_j$ and let $X_{t+1}(k) = X_t(i)$ where $k = X_t^{-1}(c_j)$ is the position of card $c_j$ in $X_t$.
- For $\ell \notin \{i, k\}$, let $X_{t+1}(\ell) = X_t(\ell)$.

Show a coupling argument to upper bound the mixing time of this Markov chain, within a constant factor of optimal is fine. Recall, the mixing time is defined to be the number of steps (from the worst initial state) to get within variation distance $\leq 1/4$ of the uniform distribution.

## 7. Approximation Algorithms

Consider the following modification to the metric uncapacitated facility location problem. Define the cost of connecting city $j$ to facility $i$ to be $c_{ij}^2$. The $c_{ij}$'s still satisfy the triangle inequality (but the new connection costs, of $c_{ij}^2$, do not). Show that factor 3 primal-dual algorithm (given below), which uses the usual LP-relaxation and dual for the facility location problem, achieves an approximation guarantee of factor 9 for this case.

**Phase 1**

Raise the dual variable $\alpha_j$ for each unconnected city $j$ uniformly at unit rate, i.e., $\alpha_j$ will grow by 1 in unit time. When $\alpha_j = c_{ij}$ for some edge $(i, j)$, the algorithm will declare this edge to be *tight*. Henceforth, dual variable $\beta_{ij}$ will be raised uniformly, and it goes towards paying for facility $i$. Each edge $(i, j)$ such that $\beta_{ij} > 0$ is declared *special*. Facility $i$ is said to be *paid for* if $\sum_j \beta_{ij} = f_i$. If so, the algorithm declares this facility *temporarily open*. Furthermore, all unconnected cities having tight edges to this facility are declared *connected* and facility $i$ is declared the *connecting witness* for each of these cities. In the future, as soon as an unconnected city $j$ gets a tight edge to $i$, $j$ will also be declared connected and $i$ will be declared the connecting witness for $j$. When all cities are connected, the first phase terminates.

**Phase 2**

Let $F_t$ denote the set of temporarily open facilities and $T$ denote the subgraph of $G$ consisting of all special edges. Let $T^2$ denote the graph that has edge $(u, v)$ iff there is a path of length at most 2 between $u$ and $v$ in $T$, and let $H$ be the subgraph of $T^2$ induced on $F_t$. Find any maximal independent set in $H$, say $I$.

All facilities in the set $I$ are declared *open*. For city $j$, define $\mathcal{F}_j = \{i \in F_t \mid (i, j) \text{ is special}\}$. Since $I$ is an independent set, at most one of the facilities in $\mathcal{F}_j$ is opened. If there is a facility $i \in \mathcal{F}_j$ that is opened, then set $\phi(j) = i$. Otherwise, consider tight edge $(i', j)$ such that $i'$ was the connecting witness for $j$. If $i' \in I$, again set $\phi(j) = i'$. In the remaining case that $i' \notin I$, let $i$ be any neighbor of $i'$ in graph $H$ such that $i \in I$. Set $\phi(j) = i$.