1. Graph Theory

Prove that there exist no simple planar triangulation T and two distinct adjacent vertices $x, y \in V(T)$ such that x and y are the only vertices of T of odd degree. Do not use the Four-Color Theorem. (Simple means no loops or parallel edges.)

2. Probability

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables. Denote

$$S_n := X_1 + \cdots + X_n.$$

(a) If $X \geq 0$ a.s. and $\mathbb{E}X = +\infty$, then

$$\frac{S_n}{n} \to +\infty \text{ as } n \to \infty \text{ a.s.}$$

(b) If $\mathbb{E}|X| = +\infty$, then

$$\limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty \text{ a.s.}$$

3. Analysis of Algorithms

Suppose we would like to find a collection of matchings such that every edge of the graph is a member of (at least) one of the matchings we selected. The goal is to pick the minimum number of matchings for our collection. Give an $\mathrm{O}(\log n)$-approximation for this problem that runs in polynomial time, where n is the number of vertices of the input graph.

4. Theory of Linear Inequalities

Let $C \subseteq R^n$ be a finitely-generated cone of full dimension and let H be an integral Hilbert basis for C. Suppose $w^T x \geq 0$ is a facet-defining inequality for C such that the components of w are relatively prime integers. Show that there exists a vector $h \in H$ such that $w^T h = 1$.

5. Combinatorial Optimization

Consider an assignment problem (AP)

$$\begin{array}{lll} \max & \sum_{i\in N}\sum_{j\in N} c_{ij}x_{ij} & \\ & \sum_{j\in N} x_{ij} = 1 & \text{for } i \in N \\ & \sum_{i\in N} x_{ij} = 1 & \text{for } j \in N \\ & x_{ij} = 0 \text{ or } 1 & \text{for } i, j \in N \end{array}$$

where $N = \{1, \ldots, n\}$.

Note: You should be able to answer (1)-(5) quickly, which is preliminary to (6). (6) is the part of this question that counts the most.

(1) State a property of the constraint matrix from which it can be deduced that all of the extreme points of the LP relaxation are integral.

(2) Let $x(I, J) = \sum_{i\in I}\sum_{j\in J} x_{ij}$ where $\emptyset \subset I, J \subseteq N$. Suppose $|I| + |J| = n + k, k \geq 1$. Prove that if x is a feasible solution, then $x(I, J) \geq k$.

(3) Now consider a constrained AP called CAP where we require $x_{2k-1,2k-1} - x_{2k,2k} = 0$ for $k = 1, \ldots, m$. Suppose we have a fractional point say

$$x = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \text{ for } n = 3, m = 1.$$

What is required of such a point to conclude that the LP relaxation of CAP is not integral? Does this point x suffice?

(4) Let $n = 5, m = 2$ in CAP. Prove that

$$x_{11} - (x_{34} + x_{43} + x_{44}) \leq 0$$

is a valid inequality.

(5) Given $\emptyset \subset I, J \subseteq N$ with $|I| + |J| = n - 1$, let

$$K = \{2r - 1 : \{2r - 1, 2r\} \subseteq I \cap J, r \leq m\}$$

$$\hat{K} = \{2r - 1 : \{2r - 1, 2r\} \subseteq (N\backslash I) \cap (N\backslash J), r \leq m\}.$$

Show that the inequality in (4) is of the form

$$\sum_{i\in K\cup\hat{K}} x_{ii} - x(I, J) \leq 0 \qquad (*)$$

with $I = J = \{3, 4\}$.

(6) Prove in general that $(*)$ is a valid inequality for CAP if $|I| + |J| = n - 1$ and $|\hat{K}| \geq 1$.

(7) Describe how you would prove that $(*)$ with the condition given in (6) is a facet of the convex hull of CAP. An actual proof is not required.

(8) Give the form of $(*)$ with the condition given in (6) for $m = 1$. In this case, the assignment constraints, side constraints, nonnegativity and $(*)$ give the convex hull. Describe how you could prove this. An actual proof is not required.

(9) Suggest an idea for separating $(*)$ with $m = 1$.

6. Algebra

Suppose G is a group of order 255. Prove that G is cyclic. (Hint: First show G has a normal subgroup of order 17 and that G has a normal cyclic subgroup of order 85.)

7. Randomized Algorithms

Recall for a pair of distributions μ and ν on a finite set Ω, their variation distance is

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{z \in \Omega} |\mu(z) - \nu(z)|$$

Consider an ergodic Markov chain on state space Ω, transition matrix P and unique stationary distribution π. Let $P^t(x, \cdot)$ denote the t-step distribution of the Markov chain starting from $x \in \Omega$. Recall the mixing time is defined to be

$$T(\epsilon) = \max_{x \in \Omega} T_x(\epsilon)$$

where

$$T_x(\epsilon) = \min \left\{ t : d_{\mathrm{TV}}(P^t(x, \cdot), \pi) \leq \epsilon \right\}$$

For the purposes of this problem we consider the following notion of intersection time. For $x, y \in \Omega$, define their intersection time as

$$T^*_{x,y} = \min \left\{ t : d_{\mathrm{TV}} \left(P^t(x, \cdot), P^t(y, \cdot) \right) \leq 1/2 \right\}$$

and let

$$T^* = \max_{x,y \in \Omega} T^*_{x,y}$$

Prove that $T(\epsilon) \leq T^* \lceil \log(1/\epsilon) \rceil$, where the log is base 2.